Uniform Convergence and Riemann Integrability

Theorem 1. *Let $f_n \in \mathcal{R}[a,b]$ be a sequence of Riemann integrable functions. Suppose the sequence converges uniformly on $[a,b]$ to a function f. Then f is Riemann integrable.*

Proof. First, there is a uniform bound B for all $\|f_n\|_\infty$, since by uniform Cauchy $\|f_n\|_\infty < \|f_N\|_\infty + 1$, for $n \geq N$ and we can take B to be $\max\{\|f_1\|_\infty, \ldots, \|f_N\|_\infty + 1\}$. Let $M = \|f\|_\infty$. Since $\|f - f_n\|_\infty \leq \epsilon$ for $n \geq N$, we know that f is bounded and the bound M satisfies $M \leq B + \epsilon$. Now take any subinterval J of $[a,b]$, and let $M_J(f) = \sup\{f(x) : x \in J\}, m_J(f) = \inf\{f(x) : x \in J\}$, etc. (the usual notation). Now that we have fixed N and ϵ, we can assert

$$-\epsilon + m_J(f_N) \leq f(x) \leq M_J(f_N) + \epsilon,$$

for any $x \in J$ and this holds for any $J \subset [a,b]$. (This works for all J, with no need to adjust N or ϵ.) This implies

$$-\epsilon + m_J(f_N) \leq m_J(f) \leq M_J(f) \leq M_J(f_N) + \epsilon.$$

Now let $\mathcal{P}$ be a partition of $[a,b]$, let J be any of the intervals in the partition, multiply by $|J|$ and sum to get

$$-\epsilon(b-a) + s_{\mathcal{P}}(f_N) \leq s_{\mathcal{P}}(f) \leq S_{\mathcal{P}}(f) \leq S_{\mathcal{P}}(f_N) + \epsilon(b-a).$$

At this point we choose the partition $\mathcal{P}$ so that for this fixed N

$$S_{\mathcal{P}}(f_N) - s_{\mathcal{P}}(f_N) \leq \epsilon.$$

With this choice of $\mathcal{P}$,

$$S_{\mathcal{P}}(f) - s_{\mathcal{P}}(f) \leq \epsilon(1 + 2(b-a)).$$

□