The Remarkable Theorem of Levy and Steinitz
Author(s): Peter Rosenthal
Source: *The American Mathematical Monthly,* Vol. 94, No. 4 (Apr., 1987), pp. 342-351
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2323094
Accessed: 31-01-2017 23:06 UTC

The Remarkable Theorem of Lévy and Steinitz

PETER ROSENTHAL, *University of Toronto*

The author received his Ph.D. from the University of Michigan in 1967, under the supervision of Paul Halmos. His main mathematical interest has been in the theory of bounded linear operators on Hilbert space, especially in results related to invariant subspaces.

1. Introduction. Everyone knows Riemann's theorem that a conditionally convergent series of real numbers can be rearranged to sum to any real number. An alternate formulation is the following: the set of all sums of rearrangements of a given series of real numbers is the empty set, a single point, or the entire real line.

What is the corresponding theorem for series of complex numbers?

Our informal survey has shown that surprisingly few mathematicians know the answer to this question, although the answer is a very natural one that was published more than eighty years ago.

The theorem is the following: *the set of all sums of rearrangements of a given series of complex numbers is the empty set, a single point, a line in the complex plane, or the whole complex plane.*

The analogue holds in *n* dimensions:

THE LÉVY-STEINITZ THEOREM. *The set of all sums of rearrangments of a given series of vectors in a finite-dimensional real Euclidean space is either the empty set or a translate of a subspace (i.e., a set of the form* $v + M$, *where* v *is a given vector and* M *is a linear subspace).*

Since a finite-dimensional complex vector space is a real vector space of twice the dimension, the Lévy-Steinitz Theorem implies that the set of rearrangements of a series in a complex Euclidean space is the empty set or a translate of a real subspace.

The theorem was first proven by P. Lévy [4] in 1905. In 1913, Steinitz [6] pointed out that Lévy's proof was incomplete, especially in the higher-dimensional cases. Steinitz [6] filled the gap in Lévy's proof and also found an entirely different approach.

The purpose of this article is to make this beautiful result more widely known.

We present Steinitz' approach, as modified by Gross [1]. The main reason that this theorem is not better known is that the difficulty of the proof seems to be out of

342

This content downloaded from 108.179.181.50 on Tue, 31 Jan 2017 23:06:01 UTC
All use subject to http://about.jstor.org/terms

proportion to the result. We have endeavored to divide the proof into easily-digested pieces with the hope of making it both accessible and interesting.

We begin with the "Polygonal Confinement Theorem" as proven by Gross [1]; this says that an arbitrarily large but finite set of vectors, each of length less than one, which sums to 0, can be rearranged so that none of the partial sums is more than a certain constant which depends only on the dimension of the vector space.

In section 3 we discuss "The Rearrangement Theorem," which states that some rearrangement of a series converges to S if a subsequence of the sequence of partial sums of the series converges to S and the sequence of terms converges to 0. This theorem, which is a consequence of the Polygonal Confinement Theorem, is surely of interest in its own right.

We present the Lévy-Steinitz Theorem in section 4. In section 5 we briefly discuss certain related results and references.

I was told of the Lévy-Steinitz Theorem by Israel Halperin. The first few times that he started to explain the proof to me, I didn't listen; I assumed that I could prove the theorem in some easier way. Finally, after I realized I couldn't prove it, I let him describe the proof. The exposition that follows is mainly based on these private lectures, for which I am extremely grateful.

2. The Polygonal Confinement Theorem.

In the Steinitz-Gross proof of the Lévy-Steinitz Theorem the basic technical lemma is the following.

THE POLYGONAL CONFINEMENT THEOREM ([6], [1]).   *For each dimension n there is a constant C_n such that whenever $\{v_i: i = 1, \ldots, m\}$ is a finite family of vectors in $\mathbf{R}^n$ which sums to 0 and satisfies $\|v_i\| \leqslant 1$ for all i, there is a permutation P of $(2, \ldots, m)$ with the property that*

$$\left\| v_1 + \sum_{i=2}^{j} v_{P(i)} \right\| \leqslant C_n$$

for every j. Moreover, we can take $C_1 = 1$ and $C_n \leqslant \sqrt{4C_{n-1}^2 + 1}$ for every n.

Proof. The case $n = 1$ is easy. If, for example, $v_1 > 0$, we can choose $P(2)$ so that $v_{P(2)} < 0$, and keep choosing negative v's until the sum of all the chosen vectors becomes negative. Then choose the next v to be positive, and keep choosing positive v's until the sum of all the chosen vectors becomes positive. Continue in this manner until all the v's are used. Since $|v_i| \leqslant 1$ for all i, it is clear that each partial sum in this arrangement is within 1 of 0. Hence, $C_1 = 1$.

The general case is proven by induction. Assume that $n > 1$ and that C_{n-1} is known to be finite, and consider a collection $\{v_i\}$ of vectors satisfying the hypotheses.

Since $\{v_i\}$ is finite there are a finite number of possible partial sums of the v's that begin with v_1; let L be such a partial sum with maximal length among all such

partial sums. Then $L = v_1 + u_1 + \cdots + u_s$, where $\{u_1, \ldots, u_s\} \subset \{v_i\}$. Let $\{w_1, \ldots, w_t\}$ denote the other v's, so that $L + w_1 + \cdots + w_t = 0$.

We use the notation $(u|v)$ to denote the Euclidean inner product of u and v.

We begin with a proof that the $\{u_i\}$ point in the same general direction as L, while the $\{w_i\}$ point in the opposite direction; (a diagram makes this very plausible).

Claim (a): $(u_i|L) \geq 0$ for all i.

To see this, suppose that $(u_i|L) < 0$, for some i. Then

$$\left((L - u_i) \Big| \frac{L}{\|L\|}\right) = \|L\| - \frac{1}{\|L\|}(u_i|L) > \|L\|,$$

so $\|L - u_i\| > \|L\|$, which contradicts the assumption that L is a longest such partial sum.

Claim (b): $(v_1|L) \geq 0$.

For if $(v_1|L) < 0$, then

$$\left(\frac{-L}{\|L\|} \Big| (v_1 + w_1 + \cdots + w_t)\right) = \left(\frac{-L}{\|L\|} \Big| (v_1 - L)\right) = \|L\| - \frac{1}{\|L\|}(L|v_1) > \|L\|,$$

so $v_1 + w_1 + \cdots + w_t$ would be a longer partial sum than L.

Claim (c): $(w_i|L) \leq 0$ for all i. For if there was an i with $(w_i|L) > 0$, then

$$\left((L + w_i) \Big| \frac{L}{\|L\|}\right) = \|L\| + \frac{(w_i|L)}{\|L\|} > \|L\|,$$

and, therefore, $\|L + w_i\| > \|L\|$. But $\|L + w_i\|$ is the length of a partial sum of the required kind. Thus this would also contradict $\|L\|$ being the longest length of such a partial sum.

We use the inductive hypothesis in the $(n - 1)$-dimensional space

$$L^\perp = \{v \in \mathbf{R}^n : (v|L) = 0\}.$$

We let v' denote the component of a vector v in $L^\perp$; i.e.,

$$v' = v - \frac{(v|L)}{\|L\|^2}L.$$

Then $L = v_1 + u_1 + \cdots + u_s$ implies $v_1' + u_1' + \cdots + u_s' = 0$. For a similar reason, $w_1' + \cdots + w_t' = 0$. By the inductive hypothesis, there exists a permutation Q of $(1, \ldots, s)$ such that

$$\left\| v_1' + \sum_{i=1}^{j} u'_{Q(i)} \right\| \leq C_{n-1} \quad \text{for } j = 1, \ldots, s,$$

and there exists a permutation R of $(2, \ldots, t)$ such that

$$\left\| w_1' + \sum_{i=2}^{j} w_{R(i)}' \right\| \leqslant C_{n-1} \quad \text{for } j = 2, \ldots, t.$$

Define $R(1) = 1$.

Now the idea is to keep the above orders within the u's and the w's (which will keep the components in $L^{\perp}$ of partial sums from being too large) and alternately "feed in" u's and w's to keep the components along L of length at most 1 (as in the proof of the case $n = 1$).

More precisely, since $(v_1|L) \geqslant 0$ and $(w_i|L) \leqslant 0$, we can choose a smallest r, say r_1, such that

$$(v_1|L) + \sum_{i=1}^{r_1} \left(w_{R(i)}|L \right) \leqslant 0.$$

Then we choose a smallest s_1 such that

$$(v_1|L) + \sum_{i=1}^{r_1} \left(w_{R(i)}|L \right) + \sum_{i=1}^{s_1} \left(u_{Q(i)}|L \right) \geqslant 0.$$

Then a smallest r_2 such that

$$(v_1|L) + \sum_{i=1}^{r_1} \left(w_{R(i)}|L \right) + \sum_{i=1}^{s_1} \left(u_{Q(i)}|L \right) + \sum_{i=r_1+1}^{r_2} \left(w_{R(i)}|L \right) \leqslant 0.$$

And so on. Arrange the vectors $\{v_i\}$ in the order

$$\left(v_1, w_{R(1)}, \ldots, w_{R(r_1)}, u_{Q(1)}, \ldots, u_{Q(s_1)}, w_{R(r_1+1)}, \ldots, w_{R(r_2)}, \ldots \right).$$

In this arrangement, clearly the components along L of each partial sum have norm at most 1. The choice of the arrangements Q and R by the inductive hypothesis insures that the components orthogonal to L of the partial sums have norms at most $C_{n-1} + C_{n-1}$. Hence, the norm of each partial sum is at most $\sqrt{(2C_{n-1})^2 + 1}$. This completes the proof.

3. The Rearrangement Theorem.

The Rearrangement Theorem is a crucial ingredient of Steinitz' proof of the Lévy-Steinitz Theorem and is also of independent interest.

For the proof of the Rearrangement Theorem it is convenient to isolate the following consequence of the Polygonal Confinement Theorem.

LEMMA 1. *If $\{v_i: i = 1, \ldots, m\} \subset \mathbf{R}^n$ and $\|\sum_{i=1}^{m} v_i\| \leqslant \varepsilon, \|v_i\| \leqslant \varepsilon$ for all i, then there is a permutation P of $(1, \ldots, m)$ such that*

$$\|v_{P(1)} + v_{P(2)} + \cdots + v_{P(r)}\| \leqslant \varepsilon(C_n + 1)$$

for $1 \leqslant r \leqslant m$.

Proof. Define $v_{m+1} = -v_1 - \cdots - v_m$ so that $\sum_{i=1}^{m+1} v_i = 0$. By the Polygonal Confinement Theorem, there is a permutation P of $(2, \ldots, m+1)$ such that

$$\left\| \frac{1}{\varepsilon} v_1 + \sum_{i=2}^{r} \frac{1}{\varepsilon} v_{P(i)} \right\| \leqslant C_n$$

for all r.

Then $\|v_1 + \sum_{i=2}^{r} v_{P(i)}\| \leqslant \varepsilon C_n$ for all r. Let $P(1) = 1$.

Now order the $\{v_i\}$ according to P, but omit v_{m+1}; since $\|v_{m+1}\| \leqslant \varepsilon$ this changes the norms of the partial sums by at most ε. Hence in this arrangement all the partial sums have norm at most $\varepsilon C_n + \varepsilon$. This proves the Lemma.

THE REARRANGEMENT THEOREM. *In* $\mathbf{R}^n$, *if a subsequence of the sequence of partial sums of a series of vectors converges to S, and if the sequence of terms of the series converges to* 0, *then there is a rearrangement of the series that sums to S.*

Proof. Let $\{v_i\}_{i=1}^{\infty}$ be a sequence of vectors in $\mathbf{R}^n$. For each m let $S_m = \sum_{i=1}^{m} v_i$. We assume that $\{S_{m_k}\} \to S$ for some subsequence $\{S_{m_k}\}$, and we must show how to rearrange the $\{v_i\}$ so that the entire sequence of partial sums converges to S. The idea is to use Lemma 1 to obtain rearrangements of each of the families $(v_{m_k+1}, \ldots, v_{m_{k+1}-1})$ so that all the partial sums of these families are small. Then S_m is close to S_{m_k} if m is between m_k and m_{k+1}.

This can be stated as follows. Let $\delta_k = \|S_{m_k} - S\|$; then $\{\delta_k\} \to 0$. Now

$$\left\| \sum_{i=m_k+1}^{m_{k+1}-1} v_i \right\| = \left\| \sum_{i=1}^{m_{k+1}} v_i - \sum_{i=1}^{m_k} v_i - v_{m_{k+1}} \right\| < \delta_{k+1} + \delta_k + \|v_{m_{k+1}}\|.$$

For each k let

$$\varepsilon_k = \max\{\delta_{k+1} + \delta_k, \sup\{\|v_i\|: i \geqslant m_k\}\}.$$

Then $\{\varepsilon_k\} \to 0$, and

$$\left\| \sum_{i=m_k+1}^{m_{k+1}-1} v_i \right\| < 2\varepsilon_k.$$

By Lemma 1, for each k there is a permutation P_k of $(m_k + 1, \ldots, m_{k+1} - 1)$ such that

$$\left\| \sum_{i=m_k+1}^{r} v_{P_k(i)} \right\| \leqslant 2\varepsilon_k(C_n + 1)$$

for $r = m_k + 1, \ldots, m_{k+1} - 1$.

Now arrange the $\{v_i\}$ as follows. Keep v_{m_k} in position m_k for each k. Then order the v_i for $(m_k + 1) \leqslant i \leqslant (m_{k+1} - 1)$ according to P_k. In this arrangement, if $m_k + 1 \leqslant m \leqslant m_{k+1} - 1$ then $S_m - S_{m_k}$ is a sum of the form $\sum_{i=m_k+1}^{m} v_{P_k(i)}$ with $m < m_{k+1}$, and hence has norm at most $2\varepsilon_k(C_n + 1)$. Since $\{S_{m_k}\} \to S$ and $\{\varepsilon_k\} \to 0$, it follows that $\{S_m\} \to S$.

4. The Lévy-Steinitz Theorem.

To prove the Lévy-Steinitz Theorem we will need another consequence of the Polygonal Confinement Theorem in addition to the Rearrangement Theorem.

LEMMA 2. *If* $\{v_i\}_{i=1}^m \subset \mathbf{R}^n$, $w = \sum_{i=1}^m v_i$, $0 < t < 1$, *and* $\|v_i\| \leqslant \varepsilon$ *for all* i, *then either* $\|v_1 - tw\| \leqslant \varepsilon\sqrt{C_{n-1}^2 + 1}$ *or there is a permutation* P *of* $(2, \ldots, m)$ *and an* r *between 2 and m such that* $\|v_1 + \sum_{i=2}^r v_{P(i)} - tw\| \leqslant \varepsilon\sqrt{C_{n-1}^2 + 1}$.

Proof. Suppose $w \neq 0$ (otherwise the result is trivial). Consider first the case $n = 1$. By multiplying through by -1 if necessary, we can assume that $w > 0$; let s denote the smallest i such that

$$v_1 + v_2 + \cdots + v_i > tw.$$

Since

$$v_1 + v_2 + \cdots + v_{s-1} \leqslant tw$$

and $|v_s| \leqslant \varepsilon$, it follows that

$$|v_1 + v_2 + \cdots + v_s - tw| \leqslant \varepsilon.$$

Thus in the case $n = 1$ the Lemma holds with $C_{n-1} = C_0$ being defined to be 0. Note also that, in the case $n = 1$, no rearranging is necessary to get an appropriate partial sum.

Now consider the general case of $\mathbf{R}^n$ for $n > 1$. Since $w = \sum_{i=1}^m v_i$, the projections $\{v_i'\}$ of the $\{v_i\}$ onto $\{w\}^\perp$ add up to 0. Since $\|v_i\| \leqslant \varepsilon$ for all i, the Polygonal Confinement Theorem yields a permutation P of $(2, \ldots, m)$ such that

$$\left\| \frac{1}{\varepsilon}v_1' + \frac{1}{\varepsilon}v_{P(2)}' + \cdots + \frac{1}{\varepsilon}v_{P(j)}' \right\| \leqslant C_{n-1}$$

for $j = 2, \ldots, m$.

Also,

$$\left(v_1 \Big| \frac{w}{\|w\|}\right) + \left(v_{P(2)} \Big| \frac{w}{\|w\|}\right) + \cdots + \left(v_{P(m)} \Big| \frac{w}{\|w\|}\right) = \|w\|,$$

and $|(v_i|w)/\|w\|| \leqslant \varepsilon$ for all i. Hence, the case $n = 1$ yields an r such that

$$\left| \left(v_1 \Big| \frac{w}{\|w\|}\right) + \left(v_{P(2)} \Big| \frac{w}{\|w\|}\right) + \cdots + \left(v_{P(r)} \Big| \frac{w}{\|w\|}\right) - t\|w\| \right| \leqslant \varepsilon.$$

The bounds on the components yield a bound on the vector, so

$$\|v_1 + v_{P(2)} + \cdots + v_{P(r)} - tw\|^2 \leqslant \varepsilon^2 C_{n-1}^2 + \varepsilon^2,$$

which is the Lemma.

Now we can finally prove the main theorem.

THE LÉVY-STEINITZ THEOREM ([4], [6]). *The set of all sums of rearrangements of a given series of vectors in* $\mathbf{R}^n$ *is either the empty set or a translate of a subspace.*

Proof. Let S denote the set of all sums of convergent rearrangements of the series $\sum_{i=1}^{\infty} v_i$. Suppose S is not empty. By replacing v_1 by $v_1 - v$, where v is any element of S, we can assume that $0 \in S$. We must show that S is a subspace.

The proof will require rearranging the series a number of times. The outline of the proof that 0, s_1 and s_2 in S implies $s_1 + s_2$ is in S is the following. We choose a sequence $\{\varepsilon_m\}$ of positive numbers that converges to 0. We form a partial sum of $\sum v_i$, in some order, that is within ε_1 of s_1. Then we construct a partial sum that contains all the vectors we have already used and that is within ε_1 of 0, then a partial sum containing all the vectors already used that lies within ε_1 of s_2, then one within ε_2 of s_1, within ε_2 of 0, ε_2 of s_2, and so on. The vectors used between a sum close to 0 and the next sum close to s_2 approximately add up to s_2. Interchanging them with those between the preceeding sum close to s_1 and the sum close to 0 produces partial sums close to $s_1 + s_2$. The Rearrangement Theorem finishes the proof.

We now present the details. Let $\{\varepsilon_m\}$ be a sequence of positive numbers that converges to 0. Since an arrangement converges to s_1, there exists a finite set I_1 of positive integers such that $1 \in I_1$ and $\|\sum_{i \in I_1} v_i - s_1\| < \varepsilon_1$. Since an arrangement converges to 0, there is a finite set J_1 of positive integers such that $J_1 \supset I_1$ and $\|\sum_{i \in J_1} v_i - 0\| < \varepsilon_1$, and a set $K_1 \supset J_1$ such that $\|\sum_{i \in K_1} v_i - s_2\| < \varepsilon_1$. There is also a set I_2 containing both K_1 and $\{2\}$ such that $\|\sum_{i \in I_2} v_i - s_1\| < \varepsilon_2$. And so on. That is, we inductively construct sets I_m, J_m, and K_m of positive integers such that

$$\{1, \ldots, m-1\} \subset K_{m-1} \subset I_m \subset J_m \subset K_m,$$

$$\left\| \sum_{i \in I_m} v_i - s_1 \right\| < \varepsilon_m, \qquad \left\| \sum_{i \in J_m} v_i - 0 \right\| < \varepsilon_m, \qquad \text{and} \qquad \left\| \sum_{i \in K_m} v_i - s_2 \right\| < \varepsilon_m.$$

For each m, starting at $m = 1$, arrange the indices in J_m so that those in I_m come at the beginning, and then arrange the indices in K_m so that those in J_m come at the beginning. Then arrange the indices of I_{m+1} so that those of K_m come at the beginning. Thus there is a permutation P of the set of positive integers and increasing sequences $\{i_m\}$, $\{j_m\}$, $\{k_m\}$ such that $i_m < j_m < k_m < i_{m+1}$, and

$$\left\| \sum_{i=1}^{i_m} v_{P(i)} - s_1 \right\| < \varepsilon_m, \qquad \left\| \sum_{j=1}^{j_m} v_{P(j)} \right\| < \varepsilon_m, \qquad \left\| \sum_{k=1}^{k_m} v_{P(k)} - s_2 \right\| < \varepsilon_m$$

for each m.

Note that

$$\left\| \sum_{i=j_m+1}^{k_m} v_{P(i)} - s_2 \right\| = \left\| \sum_{i=1}^{k_m} v_{P(i)} - \sum_{j=1}^{j_m} v_{P(j)} - s_2 \right\| < \varepsilon_m + \varepsilon_m.$$

It follows that

$$\left\| \sum_{i=1}^{i_m} v_{P(i)} + \sum_{i=j_m+1}^{k_m} v_{P(i)} - (s_1 + s_2) \right\| < 3\varepsilon_m.$$

For each m, rearrange the vectors in $\{v_{P(i)}: i = i_m, \ldots, k_m\}$ by interchanging the vectors $\{v_{P(i)}: i = i_m + 1, \ldots, j_m\}$ with the vectors $\{v_{P(i)}: i = j_m + 1, \ldots, k_m\}$. In this new arrangement, the above shows that there is a subsequence of the sequence of partial sums that converges to $s_1 + s_2$. Since we are assuming $S \neq \varnothing$, $\{v_{P(i)}\} \to 0$, so the Rearrangement Theorem implies that there is another arrangement that converges to $s_1 + s_2$. Therefore, $(s_1 + s_2) \in S$.

It remains to be shown that $s \in S$ implies $ts \in S$ for all real t. The additivity of S implies this for t a positive integer, so it suffices to consider the cases $t \in (0, 1)$ and $t = -1$.

We start with the arrangement P used above to show the additivity of S. Fix $t \in (0, 1)$. We use Lemma 2. As shown above,

$$\left\| \sum_{i=j_m+1}^{k_m} v_{P(i)} - s_2 \right\| < 2\varepsilon_m$$

for each m. Let $\delta_m = \sup\{\|v_{P(i)}\|: i = j_m + 1, \ldots, k_m\}$, and let

$$u_m = \sum_{i=j_m+1}^{k_m} v_{P(i)} - s_2.$$

By Lemma 2, there is a permutation Q_m of $\{P(j_m + 1), \ldots, P(k_m)\}$ and an r_m so that

$$\left\| \sum_{i=j_m+1}^{r_m} v_{Q_m(P(i))} - t(s_2 + u_m) \right\| \leq M\delta_m, \quad \text{where } M = \sqrt{C_{n-1}^2 + 1}.$$

Then

$$\left\| \sum_{i=j_m+1}^{r_m} v_{Q_m(P(i))} - ts_2 \right\| < M\delta_m + 2\varepsilon_m.$$

Now

$$\left\| \sum_{i=1}^{j_m} v_{P(i)} + \sum_{i=j_m+1}^{r_m} v_{Q_m(P(i))} - ts_2 \right\| < M\delta_m + 3\varepsilon_m,$$

so in this arrangement a subsequence of the sequence of partial sums converges to ts_2. The Rearrangement Theorem yields $ts_2 \in S$.

It only remains to be shown that $-s_2 \in S$. But

$$\left\| \sum_{i=1}^{j_{m+1}} v_{P(i)} - \sum_{i=1}^{k_m} v_{P(i)} - (0 - s_2) \right\| < \varepsilon_{m+1} + \varepsilon_m,$$

so

$$\left\| \sum_{i=k_m+1}^{j_{m+1}} v_{P(i)} - (-s_2) \right\| < \varepsilon_{m+1} + \varepsilon_m.$$

Then

$$\left\| \sum_{j=1}^{j_m} v_{P(i)} + \sum_{i=k_m+1}^{j_{m+1}} v_{P(i)} - (-s_2) \right\| < \varepsilon_{m+1} + 2\varepsilon_m,$$

and there is an arrangement with a subsequence of the sequence of partial sums converging to $-s_2$. By the Rearrangement Theorem, $(-s_2) \in S$.

5. Additional Remarks.

There are several natural questions related to the Lévy-Steinitz Theorem.

1) Can every translate of every subspace actually occur as a set of sums of rearrangements of a series?

The answer to this question is easily seen to be "yes". For if M is any subspace and v is any vector, let $\{e_1, \ldots, e_m\}$ be a basis for M and let $\{x_i\}$ be any conditionally convergent sequence of real numbers (e.g., let $x_i = (-1)^i/i$). Then Riemann's theorem clearly implies that the set of all sums obtained from rearrangements of the vectors $\{v, x_i e_j : j = 1, \ldots, m; i = 1, 2, 3, \ldots\}$ is $v + M$.

2) What conditions on $\{v_i\}$ determine whether the set of sums is empty, is a translate of a proper subspace, or is all of $\mathbf{R}^n$?

Both Lévy's [4] and Steinitz' [6] approach to the theorem yield an answer to this question. First note that the set of sums is empty unless $\{v_i\} \to 0$, and, for each vector w, the sums $\sum_{i=1}^{\infty}(v_i|w)^+$ and $\sum_{i=1}^{\infty}(v_i|w)^-$ are either both finite or infinite (where $(v_i|w)^+$ is 0 if $(v_i|w)$ is negative and is $(v_i|w)$ otherwise, and $(v_i|w)^-$ is 0 if $(v_i|w)$ is positive and is $-(v_i|w)$ otherwise).

If both the above conditions are satisfied then it can be shown that arrangements of $\sum_{i=1}^{\infty} v_i$ do converge.

If there is no absolute convergence in any direction (i.e., if, for each w, both $\sum_{i=1}^{\infty}(v_i|w)^+$ and $\sum_{i=1}^{\infty}(v_i|w)^-$ are infinite), then the set of sums is the entire space $\mathbf{R}^n$. If there are vectors w other than 0 for which the above sums are both finite, then the set of sums of rearrangements is $v + M$, where v is any sum and M is the orthocomplement of the set of w's such that these sums are both finite.

The above-described strengthening of what we have called the Lévy-Steinitz Theorem appears in the original papers of Lévy [4], Steinitz [6] and Gross [1]. It is also treated in [2].

3) What is the situation for other topological vector spaces?

Since all finite-dimensional topological vector spaces of the same dimension are isomorphic, the Lévy-Steinitz Theorem holds in any finite-dimensional space.

In Hilbert space there are counterexamples, the first having been found by Marcinkiewicz ([5, p. 106]). This counterexample can be imbedded in many Banach spaces as well. The theorem does hold in the topological vector space of all sequences [3]. An exposition of these results, as well as a variation of Lévy's approach, can be found in [2].

REFERENCES

1. W. Gross, Bedingt Konvergente Reihen, Monat. Math. Physik., 28 (1917), 221–237.
2. Israel Halperin, Sums of a Series, Permitting Rearrangements, C. R. Math. Rep. Acad. Sci. Canada, VIII (1986), 87–102.
3. Y. Katznelson and O. C. McGehee, Conditionally convergent series in R^{∞}, Mich. Math. J., 21, (1974), 97–106.
4. P. Lévy, Sur les séries semi-convergentes, Nouv. Ann. d. Math., 64 (1905), 506–511.
5. R. D. Mauldin, editor, The Scottish Book, Birkhauser, Boston, 1981.
6. E. Steinitz, Bedingt Konvergente Reihen and Konvexe Systeme, J. f. Math., 143 (1913), 128–175.

LETTERS TO THE EDITOR

Editor,

In a recent article in this MONTHLY [1], Stephen L. Campbell has given a proof, different from the historically well-known Cantor diagonalisation process, of the countability of the set of rational numbers.

His method is similar in spirit to that given in [2], where a Gödel index $2^n 3^p 5^q$ is constructed to correspond to the rational number $(-1)^n p/q$, where $n = 0$ or 1 depending on whether the rational number is positive or negative, respectively. This method is easily generalized to cover the case of the set of polynomials with rational coefficients as well. In my opinion, this method is more elementary.

REFERENCES

1. Stephen L. Campbell, Countability of Sets, this MONTHLY, 93 (1986) 480–481.
2. Howard Eves and Carroll V. Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics (revised edition), Holt, Rinehart and Winston, NY (1966), p. 338.

P.W. Epasinghe
Department of Mathematics
University of Colombo
Sri Lanka