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A Geometric Theory of Surface Area

Part I: Non-parametric Surfaces

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With 2 Figures

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Introduction: In 1902 LEBESGUE [2] proposed that surface area be defined as the infimum of the limit inferiors of sequences of areas of piecewise linear functions which converge uniformly to the given surface. In 1910 ZOARD DE GEÖCZE [3] published the conjecture that, without loss in generality, the sequences of piecewise linear functions may be restricted to polyhedra which are inscribed on the given surface. In 1950 MULHOLLAND [4] gave a proof of the correctness of this conjecture for non-parametric surfaces.

Despite the success of Lebesgue's definition in stimulating an enormous body of research, a number of mathematicians [5, 6, 7, 8] have sought a more simple and a more geometric definition of surface area. The aim is to discover a definite scheme for setting up sequences of suitably chosen polyhedra inscribed on the given surface such that the corresponding sequences of polyhedral areas converge to the Lebesgue area, whether this be finite or infinite. To the best of this writer's knowledge, every one of the definitions that to date have been proposed in this direction, is either less general than that of Lebesgue, or conflicts with it outside of the elementary case.

The present work provides, for non-parametric surfaces, a simple geometric definition of surface area which totally agrees with the Lebesgue definition.

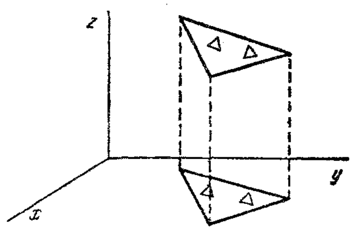
A piecewise linear function is a polyhedron no face of which is normal to the xy plane. In the present paper we do not make this restriction. We consider the set of all admissible polyhedra inscribed on the given surface (see later), subject only to the restriction that if T_1 and T_2 are any two faces of such a polyhedron then the interiors of their projections on the xy plane are disjoint.

In [9], we introduced the notion of piecewise flatness. However the discussion there was limited to continuously partially differentiable surfaces. In [10] the concept was extended to the general case of a continuous non-parametric surface. In the present paper we make a further extension of this concept as applied to non-parametric surfaces. For non-parametric surfaces, this extension leads to a general geometric theory of surface area which totally agrees with the Lebesgue theory.

The idea that underlies this theory is that, if a sequence of polyhedra $(\Pi_1, \Pi_2, \ldots)$ inscribed on a given surface S is to converge areawise to the surface S, then the orientations of the faces of the polyhedra must converge to the orientations of the pieces of S which they respectively subtend.

Neglect of this consideration by SERRET [1] a century ago led, subsequently, to the discovery of the Schwarz phenomenon. LEBESGUE [2] himself did not take explicit cognizance of this principle with the effect that his definition of surface area, in the opinion of some mathematicians, is rather distant from the simple geometric situation. And, this definition is very intractable. So intractable, in fact, that, in the words of TIBOR RADÓ [11], even such an intuitively simple proposition as that embodied in the Geöcze conjecture resisted the efforts of some of the ablest mathematicians for half a century.

In this paper we make use of inscribed triangular polyhedra every face of which has an angle which lies between a prescribed angle ϕ and $\pi - \phi$, $0 < \phi < \pi$. We refer to such polyhedra as admissible polyhedra. We also make use of triangles (inscribed in the given surface) having one angle between ϕ and $\pi - \phi$ and refer to them as admissible triangles. Since we limit our discussion to such polyhedra and to such triangles we shall, throughout, omit the qualification "admissible".



Our basic concept is that of the *deviation* of a triangle that is inscribed in the given surface.

Let T be a triangle inscribed in a given surface S. Let T_1 and T_2 be any two triangles inscribed in S and such that Proj $T_1 \subset$ Proj T and

Proj $T_2 \subset$ Proj T, i. e., the projections of T_1 and T_2 on the xy plane are subsets of that of T. Then, by $D(T)$, the deviation of T, we mean the supremum of the acute angles between the normals to T_1 and to T_2.

We now proceed formally.

1. We consider surfaces $S: z = f(x, y)$ defined and continuous on E, a closed set in the xy plane consisting of the interior and the boundary of a closed simple polygon. A triangular polyhedron Π is said to be inscribed *on* S if all the vertices of Π are in S and $\text{Proj}(\Pi) = E$. Of two such polyhedra Π_1 and Π_2, we say that Π_2 is a refinement of Π_1 if every vertex of Π_1 is a vertex of Π_2. A triangle T is said to be inscribed *in* S if all its vertices are in S.

If T is a face of a polyhedron inscribed on S, then $D(T) =$ supremum of the angles between the normals to triangles T_1 and T_2 each inscribed in S and such that Proj $(T_1 \cup T_2) \subset$ Proj T.

Definitions: $S = f(E)$ is said to be quasi-piecewise flat (qpf) if, for every $\alpha > 0$ and every $\beta > 0$, there exists a triangular polyhedron Π inscribed on S such that

a) For each of some of the faces of Π (the so-called α-regular faces), the deviation is less than α, and

b) The sum of the areas of the faces of Π whose deviations are greater than or equal to α, is less than β. We refer to these faces as α-irregular faces.

We refer to such a polyhedron as an (α, β) polyhedron.

We consider infinite sequences $(\Pi_1, \Pi_2, \ldots)$ of polyhedra inscribed on S such that the corresponding sequences $(\alpha_1, \alpha_2, \ldots)$ and $(\beta_1, \beta_2, \ldots)$ both converge to zero. Clearly, if S is qpf, such a sequence of inscribed polyhedra exists. We call it a regular sequence of inscribed polyhedra.

A regular sequence $(\Pi_1, \Pi_2, \ldots)$ of polyhedra inscribed on S is said to be strongly regular if $\sec \theta$ is bounded, θ being the acute angle between the z-axis and the normal to any face of any polyhedron in the sequence. A qpf surface is said to be strongly quasi-piecewise flat ($sqpf$) if S permits the inscription of a strongly regular sequence $(\Pi_1, \Pi_2, \ldots)$ of inscribed polyhedra.

Let Π be a polyhedron inscribed on S and let m, $m < \infty$, be an upper bound of the secant of the acute angles between the z-axis and the normals to the faces of Π. If Π^* is a refinement of Π such that m is also an upper bound of the secant of the acute angles between the z-axis and the normals to the faces of Π^*, then we refer to Π^* as a regular refinement of Π.

By $D(P)$, the deviation of $P \in E$, we mean the infimum of the set of deviations of triangles T inscribed in S such that P is in the interior of Proj T, unless P is the boundary of E, in which case, we drop the requirement of interiority (i. e., it is sufficient then that $P \in \text{Proj } T$).

Theorem 1. *Let $S = f(E)$ be sqpf. Let $(\Pi_1, \Pi_2, \ldots)$ be a strongly regular sequence of polyhedra inscribed on S. If $(A_1, A_2, \ldots)$ is the corresponding sequence of polyhedral areas, then $(A_1, A_2, \ldots)$ converges. Moreover, for all strongly regular sequences of polyhedra inscribed on S, the limit of the polyhedral areas is unique and independent of ϕ, the particular admissibility number.*

We make use of a lemma.

Lemma 1: Let Π be an (α, β) polyhedron inscribed on S. Let m, $m < \infty$, be an upper bound of sec θ. *Let Π^* be a regular refinement of Π. Let A denote the area of Π, B the sum of the areas of the projections of the regular faces of Π, and A^* the area of Π^*. Then $|A - A^*| < MB\alpha + \beta\,(1+m)$ where M is a Lipschitz constant involved in the inequality*

$$|\sec\theta_1 - \sec\theta_2| < M\,|\theta_1 - \theta_2|.$$

Proof of the lemma: sec θ is continuously differentiable on the closed interval $[0, \theta_m]$, where $\theta_m = \operatorname{arcsec} m$. Thus, sec θ is uniformly Lipschitzian on $[0, \theta_m]$.

Let A_1 denote the sum of the areas of the α-regular faces of Π and A_1^* the sum of the areas of the faces of Π^* which are subtended by the regular faces of Π. Then, one shows easily [10] that $|A_1 - A_1^*| < MB\alpha$.

Let A_2 denote the sum of the areas of the irregular faces of Π and A_2^* the sum of the areas of the faces of Π^* which are subtended by the irregular faces of Π. Then, $|A_2 - A_2^*| \leq A_2 + A_2 m < \beta\,(1 + m)$. Thus, $|A - A^*| < MB\alpha + \beta\,(1 + m)$.

We now proceed to the proof of Theorem 1.

Proof: Corresponding to the sequence $(\Pi_1, \Pi_2, \ldots)$, let us consider the sequences $(\alpha_1, \alpha_2, \ldots)$, $(\beta_1, \beta_2, \ldots)$ and $(A_1, A_2, \ldots)$, this last being the corresponding sequence of the areas of the polyhedra.

Let m, $m < \infty$, be a uniform upper bound of sec θ. Let $\varepsilon > 0$ be given. Let D denote the area of E.

There exists a positive integer N_1 such that if $n > N_1$, then

$$\alpha_n < \frac{\varepsilon}{4MD} \text{ and } \beta_n < \frac{\varepsilon}{4(1+m)}.$$

Also, there exists a positive integer N_2 such that if $n > N_2$, then every refinement of the polyhedron composed of the α_n-regular faces of Π_n is a regular refinement. Let N be the larger of N_1 and N_2.

Let $n_1 > N$ and $n_2 > N$.

Consider the union U of the vertices of the α_{n_1}-regular faces of Π_{n_1} and the vertices of the α_{n_2}-regular faces of Π_{n_2}. Let A_{n_1} and A_{n_2} denote the areas of Π_{n_1} and of Π_{n_2}, respectively. The points of U, if necessary, with the addition of a finite set of well chosen points, give an admissible polyhedron which is a common refinement of Π_{n_1} and Π_{n_2}. Let B denote the area of this polyhedron.

Then

$$| A_{n_1} - B | < MD\alpha_{n_1} + \beta_{n_1}(1 + m) \text{ and}$$
$$| A_{n_2} - B | < MD\alpha_{n_2} + \beta_{n_2}(1 + m).$$

$$| A_{n_1} - B | < MD \frac{\varepsilon}{4MD} + \frac{\varepsilon}{4(1+m)} (1 + m) = \frac{\varepsilon}{2},$$

$$| A_{n_2} - B | < MD \frac{\varepsilon}{4MD} + \frac{\varepsilon}{4(1+m)} (1 + m) = \frac{\varepsilon}{2},$$

$$| A_{n_1} - A_{n_2} | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence the sequence $(A_1, A_2, \ldots)$ converges.

One shows easily that for the set of all strongly regular sequences of polyhedra inscribed on S, the corresponding sequences of polyhedral areas converge to a unique limit and that this is independent of ϕ and m.

Theorem 2. *Let $S = f(E)$ be any continuous surface and let P be interior to E. If $D(P) = 0$, then f is continuously partially differentiable at P.*

Here, as everywhere else in this paper, the partial derivatives are taken to exist provided the corresponding plane sections of S have a tangent line at $f(P)$. As such these partials may be finite or infinite. Here, continuity of the partials at P means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if f has finite partial derivatives at P and at Q, then if the distance between P and Q is less than δ, it follows that $|f_x(P) - f_x(Q)| < \varepsilon$ and $|f_y(P) - f_y(Q)| < \varepsilon$.

We make use of an easily proved [10] lemma.

Lemma 2. Let $D(P) = 0$. Let $(T_1, T_2, \ldots)$ be an infinite sequence of triangles inscribed in S and such that for each n, P is an element of Proj (T_n). *Let the corresponding sequence $(d_1, d_2, \ldots)$ of diameters of the triangles converge to zero. Let $(V_1, V_2, \ldots)$ be the corresponding sequence of unit vectors such that each V_n is normal to T_n. Then the sequence $(V_1, V_2, \ldots)$ converges to a unique unit vector $V(P)$.*

Corollary 1: If $D(P) = 0$, then the surface S has a normal line and a tangent plane at $Q = f(P)$.

We proceed to the proof of Theorem 2.

Let C_x and C_y be, respectively, the curves of intersection of S with the planes through Q parallel to the xz plane and the yz plane.

Let $(q_1, q_2, \ldots)$ be an infinite sequence of points on C_x converging to Q. Let $(Vq_1, Vq_2, \ldots)$ be the corresponding sequence of unit vectors from Q through $q_1, q_2, q_3, \ldots$, respectively.

If the set $\{Vq_1, Vq_2, \ldots\}$ is finite then there exists a convergent subsequence of $(Vq_1, Vq_2, \ldots)$. If the set $\{Vq_1, Vq_2, \ldots\}$ is infinite, then there exists a vector limit point of the set. There exists then a subsequence of $(Vq_1, Vq_2, \ldots)$ which converges to this vector limit point. Thus, in either case, there exists a convergent subsequence.

Similarly, if $(r_1, r_2, \ldots)$ is any sequence of points on C_y converging to Q and $(Wr_1, Wr_2, \ldots)$ is the corresponding sequence of unit vectors through Q and through $(r_1, r_2, \ldots)$ respectively, then there exists a convergent subsequence of $(Wr_1, Wr_2, \ldots)$.

Suppose now that there exist two subsequences of $(Vq_1, Vq_2, \ldots)$ which converge to two distinct unit vectors. Let these be V_x^* and V_x^{**}. Let a subsequence of $(Wr_1, Wr_2, \ldots)$ converge to V_y. Then

$$V_x^* \times V_y \neq V_x^{**} \times V_y\,.$$

This contradicts the foregoing lemma.

Since the sequences $(Vq_1, Vq_2, \ldots)$ converge to a unique limit vector, it follows that $\frac{\partial f}{\partial x}$ exists at P. Similarly, $\frac{\partial f}{\partial y}$ exists at P. By a similar argument, one shows that f has a directional derivative at P in every direction.

We now show that the partial derivatives are continuous at P. We make use of the above corollary.

$D(P) = 0$. Hence, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if T is any triangle inscribed in S and Proj $T \subset N(P, \delta)$, the δ-neighborhood of P, then the acute angle between the normal to S at $Q = f(P)$ and the normal to T is less than ε.

Suppose that $P_1 \in N\left(P, \frac{\delta}{2}\right)$ and f_x and f_y both exist at P_1. Let $Q_1 = f(P_1)$. Consider Q_1X and Q_1Y, the curves of intersection of S with the planes through Q_1 and parallel to the xz and the yz planes, respectively. There exist tangent lines to Q_1X and Q_1Y. There exists a small triangle $T_1 = Q_1X_1Y_1$, where $X_1 \in Q_1X$, $X_1 \neq X$, $Y_1 \in Q_1Y$, $Y_1 \neq Y$, and Proj $T_1 \subset N(P, \delta)$. Since the acute angle between the normal to S at Q and the normal to the triangle $Q_1X_1Y_1$ is less than ε, it follows that f_x and f_y are both continuous at P.

Theorem 3. *Let $S = f(E)$ be qpf. Then, for each $\varepsilon > 0$, the set of the points P for which $D(P) \geq \varepsilon$ is of Jordan measure zero.*

Proof: Let $\varepsilon > 0$ be given. Let $\beta > 0$ be given.

There exists a finite triangulation of E such that the deviation of each of the regular faces of the corresponding inscribed polyhedron Π is less than ε and the sum of the areas of the irregular faces is less than β. The deviation at each point P which is interior to the projection of a regular face is less than ε. The deviation at points which are on the boundary of the projection of a regular face may be greater than or equal to ε, but the Jordan measure of this set is zero. The sum of the areas of the projections of the irregular face is less than β. Since $\beta > 0$ is arbitrary, the set of the points $P \in E$ for which $D(P) \geq \varepsilon$ is of Jordan measure zero.

Corollary 2: If $S = f(E)$ is qpf, then the set of the points P of E at which $D(P) \neq 0$ is of Lebesgue measure zero.

Corollary 3: If $S = f(E)$ is qpf, then f is partially differentiable on E almost everywhere (in the Lebesgue sense). Also, f is continuously partially differentiable on E almost everywhere.

The foregoing corollary gives us a necessary condition if $S = f(E)$ is to be *qpf*. This condition is not sufficient for S to be *qpf*.

Definition: Let $S = f(E)$. The function f is said to be absolutely continuous on E if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if G

is a finite set of non-overlapping triangles in E the sum of whose areas is less than δ, then the sum of the areas of the triangles inscribed in S which are subtended by the triangles in G is less than ε.

Theorem 4. *Let $S = f(E)$, where f is absolutely continuous on E. If for each $\varepsilon > 0$, the set of points P of E for which $D(P) \geq \varepsilon$ is of Jordan measure zero, then S is qpf.*

Proof: Let $\alpha > 0$ and $\beta > 0$ be given. Let δ be the δ associated with $\varepsilon = \alpha$ in the definition of absolute continuity. There exists a simple set F (i. e., the union of a finite set of closed rectangles), subset of E, which contains in its interior all the points P of E for which $D(P) \geq \frac{\varepsilon}{2}$ and whose area (i. e., the area of F) is less than δ. This set is finitely triangulable. Since E itself is finitely triangulable, it is possible to obtain a new triangulation of E which includes the triangulation of F.

Consider the set $\overline{E - F}$. For each point R of $\overline{E - F}$ there exists a small triangle T containing R as an interior point and such that $D(T^*) < \alpha$, where T^* is the triangle inscribed in S such that $\text{Proj } T^* = T$. Letting R range over all of $\overline{E - F}$ yields a covering Γ of $\overline{E - F}$. Since $\overline{E - F}$ is compact, there exists a finite subset Γ^* of Γ which covers $\overline{E - F}$. These considerations yield a triangulation of E which satisfies the definition of quasi-piecewise flatness.

Corollary 4: Let $S = f(E)$, f absolutely continuous on E. If, for each $\varepsilon > 0$, the set E_ε of the points P of E for which $D(P) \geq \varepsilon$ is of Lebesgue measure zero, then S is qpf.

Proof: E_ε is closed and bounded and so, is compact.

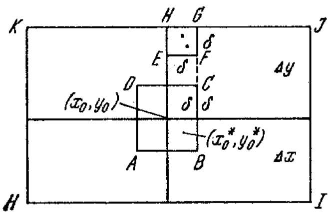
Theorem 5. *Let $S = f(E)$ be qpf. If f is continuously partially differentiable at $P \in E$, then $D(P) = 0$.*

We make use of a lemma.

Lemma 3. Let $z = f(x, y)$ be continuous on E and be partially differentiable on E, except possibly on a set of Lebesgue measure zero. Let f_x and f_y be continuous on their domains. Let f be partially differentiable at $P: (x_0, y_0) \in E$. Then, for each $\varepsilon > 0$, there exists $\tau > 0$ such that, if $|\Delta x| < \tau$, $|\Delta y| < \tau$ and $(x_0 + \Delta x, y_0 + \Delta y) \in E$, then

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) =$$
$= f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \eta\Delta x + \mu\Delta y + \overline{\Delta x}^2\,\overline{\Delta y}^2$, *where* $|\eta| < \varepsilon$ *and* $|\mu| < \varepsilon$.

Proof: We take up the case where $P:(x_0, y_0)$ is interior to E.



For given Δx and Δy, $|\Delta x| < 1$ and $|\Delta y| < 1$, with the rectangle $HIJK$ with center at $P:(x_0, y_0)$, a subset of E, the function

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y),$$

as a function of x and y, is continuous at P Hence, there exists $\delta > 0$ such that, if $Q:(x_1, y_1)$ is inside the square $ABCD$, then

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) = f(x_1 + \Delta x, y_1 + \Delta y) -$$
$$- f(x_1, y_1 + \Delta y) + \lambda, \text{ where } |\lambda| < \overline{\Delta x}^2\,\overline{\Delta y}^2.$$

Inside the square $EFGH$ there exists a point $(x_0^*, y_0^* + \Delta y)$ at which f_x and f_y exist.

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) = f(x_0^* + \Delta x, y_0^* + \Delta y) -$$
$$- f(x_0^*, y_0^* + \Delta y) + \lambda, \text{ where } |\lambda| < \overline{\Delta x}^2\,\overline{\Delta y}^2.$$

Since f_x exists at $(x_0^*, y_0^* + \Delta y)$, for each $\varepsilon > 0$, there exists $\alpha > 0$ such that, if $|\Delta x| < \alpha$, then $f(x_0^* + \Delta x, y_0^* + \Delta y) - f(x_0^*, y_0^* + \Delta y) = f_x(x_0^*, y_0^* + \Delta y)\,\Delta x + \mu\Delta x$, where $|\mu| < \varepsilon$.

Thus, $f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) = f_x(x_0^*, y_0^* + \Delta y)\Delta x + \mu\Delta x + \lambda$.

Since f_x is continuous on its domain, there exists $\beta > 0$ such that, if $|\Delta x| < \alpha$ and $|\Delta x| < \beta$, then

$$f_x(x_0^*, y_0^* + \Delta y) = f_x(x_0^*, y_0^*) + \theta, \text{ where } |\theta| < \frac{\varepsilon}{2}.$$

Thus, $f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) = f_x(x_0, y_0)\Delta x + \theta\Delta x + \mu\Delta x + \lambda$.

Since f_y exists at (x_0, y_0), there exists $\gamma > 0$ such that, if $|\Delta y| < \gamma$, then $f(x_0, y_0 + \Delta y) - f(x_0, y_0) = f_y(x_0, y_0)\Delta y + \phi\Delta y$, where $|\phi| < \frac{\varepsilon}{2}$. Hence if $|\Delta x|$ and $|\Delta y|$ are each less than τ, the lesser of α, β, and γ, then

$$f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)=f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y+(\theta+\mu)\Delta x+\phi\Delta y+\lambda,$$

where $|\theta| < \frac{\varepsilon}{2}$, $|\mu| < \frac{\varepsilon}{2}$, $|\phi| < \varepsilon$ and $|\lambda| < \overline{\Delta x}^2\,\overline{\Delta y}^2$.

To complete the proof when P is interior to E, take $\eta = \theta + \mu$, and $\mu = \phi$.

The case when P is on the boundary of E, is handled in the obvious manner.

Corollaries of the lemma:

Under the hypothesis of the above lemma,

5. *For each direction, the directional derivative of f at P exists in that direction.*

6. *The differential of f at P exists.*

7. *At P, S has a tangent plane.*

8. *For each $\varepsilon > 0$ there exists $\delta > 0$ such that, if T_1 and T_2 are any two triangles in $S(P, \delta)$ then the acute angle between the normals to T_1^* and T_2^*, triangles inscribed in S and such that* Proj $T_1^* = T_1$ *and* Proj $T_2^* = T_2$, *is less than ε.*

Theorem 5 is an immediate consequence of Corollary 9.

Corollary 9: Let $S = f(E)$ be qpf and let $P \in E$. Then $D(P) = 0$ if and only if f is continuously partially differentiable at P.

Definition: $P \in E$ is said to be an irregular point of E if for every neighborhood $N(P, \delta)$, $\sec\theta$ is unbounded, θ being the acute angle between the z-axis and the normal to a triangle T inscribed in S and such that Proj $(T) \subset N(P, \delta)$. A point $Q \in E$ is said to be regular if it is not an irregular point of E.

Corollary 10: Let $S = f(E)$ be qpf. If E does not contain any irregular points, then S is sqpf.

Proof: E is compact.

Theorem 6. *Let $S = f(E)$ be qpf and let E_i denote the set of the irregular points of E. If E_i is of Lebesgue measure zero, then there exists a regular sequence $(\Pi_1, \Pi_2, \ldots)$ of polyhedra inscribed on S such that the corresponding sequence $(A_1^*, A_2^*, \ldots)$ of the polyhedral areas converges to the Lebesgue area of S, whether this be finite or infinite.*

Proof: For every $\varepsilon > 0$, there exists a closed subpolygon $E^*_\varepsilon \subset E$ (E^*_ε is a closed set bounded by a closed polygon) such that on E^*_ε there exist no irregular points of E and area of E — area of $E^*_\varepsilon < \varepsilon$.

Let $(\Pi^*_1, \Pi^*_2, \ldots)$ be a strongly regular sequence of polyhedra inscribed on E^*_ε. By Theorem 6 of [10] the corresponding sequence of areas $(A^*_1, A^*_2, \ldots)$ converges to the Lebesgue integral $\int_{E^*_\varepsilon} \sqrt{1+f_x^2+f_y^2}$.

This, by Section 4 of [10], is precisely the Lebesgue area L_ε of $S^*_\varepsilon = f(E^*_\varepsilon)$.

Consider now a sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ converging to zero. Let $(\Pi_1, \Pi_2, \ldots)$ be a strongly regular sequence of polyhedra inscribed on $S^*_{\varepsilon_1} = f(E^*_{\varepsilon_1})$.

On $\overline{E - E^*_{\varepsilon_1}}$ there exists a finite triangulation A_2 of area less than ε_2 which contains all the irregular points of E. Let

$$E^*_{\varepsilon_2} = E^*_{\varepsilon_1} \cup \left(\overline{(E - E^*_{\varepsilon_1}) - A_2}\right).$$

With $(\Pi_{11}, \Pi_{12}, \ldots)$ already constructed, we construct the corresponding strongly regular sequence $(\Pi_{21}, \Pi_{22}, \ldots)$ of polyhedra inscribed on $E^*_{\varepsilon_2}$, seeing to it that, for each i, Π_{2i} contains Π_{1i} as a subpolyhedron. We construct the regular sequence $(\Pi_{31}, \Pi_{32}, \ldots)$ using $(\Pi_{21}, \Pi_{22}, \ldots)$ in analogous manner.

Continuing this procedure indefinitely gives us a sequence of sequence

$$\begin{aligned}
\varepsilon_1&: (\Pi_{11}, \Pi_{12}, \ldots) \\
&\quad (A^*_{11}, A^*_{12}, \ldots) \text{ converges to } \iint_{E^*_{\varepsilon_1}} = L_{\varepsilon_1} \\
\varepsilon_2&: (\Pi_{21}, \Pi_{22}, \ldots) \\
&\quad (A^*_{21}, A^*_{22}, \ldots) \text{ converges to } \iint_{E^*_{\varepsilon_2}} = L_{\varepsilon_2} \\
\varepsilon_3&: (\Pi_{31}, \Pi_{32}, \ldots) \\
&\quad (A^*_{31}, A^*_{32}, \ldots) \text{ converges to } \iint_{E^*_{\varepsilon_3}} = L_{\varepsilon_3} \\
&\vdots
\end{aligned}$$

Consider now the sequence $\left(\iint_{E^*_{\varepsilon_1}}, \iint_{E^*_{\varepsilon_2}}, \cdots\right)$. Here $\iint_{E^*_{\varepsilon_1}} \leq \iint_{E^*_{\varepsilon_2}} \leq \cdots$.

If this sequence is unbounded then, by the additivity of Lebesgue area, the Lebesgue area of S is infinite.

Now suppose that the sequence $(\iint\limits_{E^*_{\varepsilon_1}}, \iint\limits_{E^*_{\varepsilon_2}}, \ldots)$ is bounded. Then this sequence converges to a real number. Since the Lebesgue integral $\iint\limits_{E}\sqrt{1+f_x^2+f_y^2}$ exists, the sequence $(\iint\limits_{E^*_{\varepsilon_1}}, \iint\limits_{E^*_{\varepsilon_2}}, \ldots)$ converges to

$$\iint\limits_{E}\sqrt{1+f_x^2+f_y^2}.$$

We now wish to set up a regular sequence of polyhedra $(\Pi^*_1, \Pi^*_2, \ldots)$ inscribed on S such that the corresponding sequence $(A^*_1, A^*_2, \ldots)$ converges to the Lebesgue area of S.

Π^*_1 is built from Π_{11} by merely adjoining a polyhedron (having no face parallel to the z-axis) on $\overline{E - E^*_{\varepsilon_1}}$. Π^*_2 is built from Π_{22} by merely adjoining a polyhedron (having no face parallel to the z-axis) on $\overline{E-E^*_{\varepsilon_2}}$, etc. Since the sequence $(\iint\limits_{E^*_{\varepsilon_1}}, \iint\limits_{E^*_{\varepsilon_2}}, \ldots)$ converges to $\iint\limits_{E}\sqrt{1+f_x^2+f_y^2}$, the sequence $(A^*_1, A^*_2, \ldots)$ also converges to $\iint\limits_{E}\sqrt{1+f_x^2+f_y^2}$.

Since the corresponding sequence of piecewise linear functions of x and y converges uniformly to S, it follows by (4) of [10] that this limit of $(A^*_1, A^*_2, \ldots)$ is the Lebesgue area of S.

The identical procedure followed in the case where the sequence $(\iint\limits_{E^*_{\varepsilon_1}}, \iint\limits_{E^*_{\varepsilon_2}}, \ldots)$ is unbounded yields the limit ∞ which is the Lebesgue area of S.

We now consider the case where the set E_i is of positive outer Lebesgue measure. For this we have the following theorem.

Theorem 7. *Let $S = f(E)$ be qpf. If the set E_i of the irregular points of E is of positive outer Lebesgue measure, then the Lebesgue area of S is infinite.*

Proof: We make use of an easily proved lemma.

Lemma 4. At each point Q of E_i, one of the two partial derivatives $\dfrac{\partial f(Q)}{\partial x}$ and $\dfrac{\partial f(Q)}{\partial y}$ does not exist finitely (if it exists at all).

To proceed to the proof of the theorem, suppose that S is of finite Lebesgue area. Then, by a well known theorem [11], f is finitely partially differentiable almost everywhere on E. This contradicts the fact that E_i is of positive outer Lebesgue measure.

We now consider the case where $S = f(E)$ is not *qpf*. There are two subcases:

1. The set E^* of the points P of E for which $D(P) \neq 0$ is of Lebesgue measure zero. For this we have the following theorem.

Theorem 8. *Let $S = f(E)$ be not qpf. Let E^* be of Lebesgue measure zero. Then there exists a regular sequence $(\Pi_1^*, \Pi_2^*, \ldots)$ of polyhedra inscribed on S such that the corresponding sequence $(A_1^*, A_2^*, \ldots)$ of polyhedral areas converges to the Lebesgue area of S, whether this be finite or infinite.*

Proof: The proof is similar to that of Theorem 6.

We now take up case 2.

2. The set of E^* is of positive Lebesgue outer measure. For this we have the following theorem:

Theorem 9. *Let $S = f(E)$ be not qpf. Let E^* be the set of points P of E for which $D(P) \neq 0$. If E^* is of positive Lebesgue outer measure, then the Lebesgue area of S is infinite.*

Proof: We make use of two lemmas.

Lemma 5. There exists δ, $\delta > 0$, such that the set $E_\delta = \{All\ P \in E^*$ such that $D(P) > \delta\}$ is of positive outer Lebesgue measure.

Proof: Let $\delta_n = \frac{1}{n}$ for each positive integer n. Then $E^* = \bigcup_{n=1}^{\infty} E_{\delta_n}$. If each E_{δ_n} is of zero outer Lebesgue measure, then each E_{δ_n} is Lebesgue measurable and its Lebesgue measure is zero. Since Lebesgue measure is completely additive, it follows that E^* is measurable and its Lebesgue measure is zero. Hence, its outer Lebesgue measure is zero. This contradicts the hypothesis that E^* is of positive outer Lebesgue measure.

Lemma 6. At each point Q of E^* one of the two partial derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ does not exist finitely (if it exists at all).

This lemma follows easily from the following elementary geometric proposition:

For each α, $0 < \alpha \leq \frac{\pi}{2}$, there exists $\beta > 0$ such that if two planes have a dihedral angle greater than α, then either the angle between their traces on the xz plane or the angle between their traces on the yz plane is greater than β.

We now proceed to the proof of the main theorem.

Suppose that S is of finite Lebesgue area. Then, by a well-known theorem [11], f is finitely partially differentiable on E almost everywhere. This contradicts the fact that E^* is of positive Lebesgue outer measure. Hence S is of infinite Lebesgue area.

The proofs of Theorems 6—9 constitute a constructive proof of the validity of the Geöcze conjecture.

The foregoing theory constitutes a geometric theory of the area of a non-parametric surface. We have shown that, for non-parametric surfaces, this theory is totally equivalent to the analytic theory of Lebesgue. It may thus be looked upon as a geometric realization of the Lebesgue theory for such surfaces.

In this paper we have not exploited the fact that we permit our polyhedra to have some faces which are parallel to the z-axis. We shall do so in our treatment of the parametric surfaces.

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