Math 336 Sample Problems

One notebook sized page of notes (both sides) will be allowed on the test. The test will cover up to §4.5 in the text and will be comprehensive. The final exam is at 8:30 am, Monday, June 6, in the regular classroom.

1. Let $u(x, y)$, $v(x, y)$ be continuously differentiable as functions of (x, y) in a domain Ω. Let $f(z) = u(z) + iv(z)$. Suppose that for every $z_0 \in \Omega$ there is an r_0 (depending on z_0) such that
$$\int_{|z-z_0|=r} f(z)dz = 0,$$
for all r with $r < r_0$. Prove that f is analytic in Ω. Hint: Show that f satisfies the Cauchy-Riemann equations in Ω.

2. Let $D_2 = \{z : |z| < 2\}$ and $I = \{x \in \mathbf{R} : -1 \leq x \leq 1\}$. Find a bounded harmonic function u, defined in $D_2 - I$ such that u does not extend to a harmonic function defined in all of D_2.

3. Find a conformal map from the region between the two lines $y = x$ and $y = x + 2$ to the upper half plane, which sends 0 to 0.

4. Find a function, $h(x, y)$, harmonic in $\{x > 0,\ y > 0\}$, such that
$$h(x,y) = \begin{cases} 0 & \text{if } 0 < x < 2, y = 0, \\ 1 & \text{if } x > 2, y = 0, \\ 2 & \text{if } x = 0, y > 0 \end{cases}$$

5. The function
$$f(z) = \frac{1}{(z+1)(z+3)}$$
has three different Laurent series expansions in powers of z. Find them all and specify the annulus for each series.

6. Let
$$f(z) = \exp(z + \frac{1}{z})$$
Prove that
$$\mathrm{Res}[f, 0] = \sum_{k=0}^{k=\infty} \frac{1}{k!(k+1)!}$$
Carefully justify all steps in your computation.

7. Suppose that u is a harmonic function and v is its conjugate harmonic function. Prove that uv is harmonic.

8. Let $f_n(z)$ be a sequence of analytic functions defined on an open connected set Ω that converges uniformly on all compact subsets of Ω. Use Morera's theorem to prove that the limit function is analytic.

9. Let $u(z)$ be harmonic in $\{z : 0 < |z| < 1\}$. Let $P = \int_{|z|=r} \frac{\partial u}{\partial n} ds$, where $0 < r < 1$. Show that P does not depend on r. Prove that
$$u(z) = \frac{P}{2\pi} \log|z| + \mathrm{Re}(f(z))$$
where f is analytic in $\{z : 0 < |z| < 1\}$.

10. Find the Green's function for the region $\{z : |z-2| < 3\}$ with pole at 4.

11. There may be homework problems or example problems from the text on the final.