Math 336 Sample Problems

One notebook sized page of notes will be allowed on the test. The test will cover up to §4.4 in the text (excluding those sections for which there was no homework).

1. p. 195, # 11, # 14

2. p. 253, # 15, # 16

3. Let $u(z)$ be harmonic on $D = \{|z| < 1\}$ and continuous on the closure of D. Use the Poisson integral formula to prove
$$\frac{1-r}{1+r}u(0) \leq u(z) \leq \frac{1+r}{1-r}u(0),$$
where $|z| = r$.

4. Let C be the open arc of the unit circle in the first quadrant. Find a bounded function u, which is harmonic on $\{z : |z| < 1\}$, with
$$\lim_{z \to \zeta} u(z) = \begin{cases} 1 \text{ if } \zeta \in C \\ 0 \text{ if } \zeta \notin \overline{C}, \ |\zeta| = 1 \end{cases}$$
Find an unbounded harmonic function with the same properties.

5. Is there an analytic function f such that $|f(z)| < 1$ for $|z| < 1$, with $f(0) = \dfrac{1}{2}$ and $f'(0) = 1$? Hint: Use Schwarz's Lemma.

6. Using Rouché's theorem, show that $z^5 + 5z^3 + z - 2$ has three roots in the set $\{z : |z| < 1\}$.

7. Let f be analytic on $D = \{z < 1\}$. Suppose f is $1-1$ on $D - \{0\}$. Prove that f is $1-1$ on D.

8. Let $u(z)$ be harmonic in all of $\mathbf{C}$. Suppose $|u(z)| \leq c|z|^n$ for some positive constant c. Prove that u is the real part of a complex polynomial of degree at most n. Hint: Use

$$u(z) = \mathrm{Re}\left\{ \left(\frac{1}{2\pi i}\right) \int_{|\zeta|=r} u(\zeta) \left(\frac{\zeta+z}{\zeta-z}\right) \frac{d\zeta}{\zeta} \right\},$$

if $|z| < r$.

9. There may be homework problems, example problems from the text, or requests for statements of theorems on the midterm.