A Surprising Application of Non-Euclidean Geometry

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Contents

1 Introduction

Plane geometry is based upon the world in which we live—that is, it is based upon Euclid's 5 postulates:

- Postulate 1: Two points determine a line.
- Postulate 2: A segment can be extended into a line.
- Postulate 3: A line segment determines a circle, with the segment as the radius.
- Postulate 4: All right angles are congruent.
- Postulate 5: If a line falling on a pair of lines makes the interior angles on the same side less than two right angles, the pair of lines meet on the side of the single line on which the interior angles are less than two right angles.

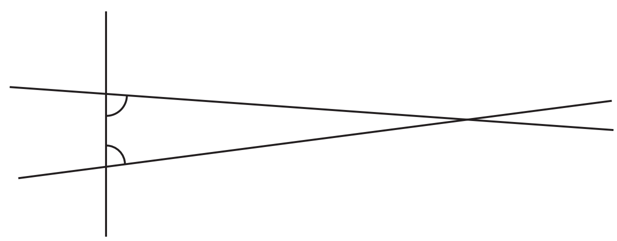

Figure 1: The Euclidean Parallel Postulate

The most lengthy postulate, Postulate 5, has caused considerable controversy among mathematicians. Is it necessary? Does assuming its converse give contradiction? John McLeary derives an elegant result obtained from ignoring the infamous "parallel postulate". By combining the subjects of non-Euclidean geometry, trigonometry, and infinite fractions, he proves the irrationality of π and e. The following is derived from and contains parts of his paper [3].

2 Geometry

One example of non-Euclidean Geometry is the geometry of a sphere—if we take the great circles to be the lines, then there are *no* parallel lines. Thus, we begin with a definition regarding mappings from spheres.

Definition 2.1. *Let the sphere of radius R centered at the origin be denoted S^2_R, and let U be an open subset of S^2_R. Then a* map projection *from S^2_R to the plane is a mapping $X : U \to \mathbb{R}^2$ that is differentiable, one-to-one, and has a differentiable inverse.*

2.1 Arclength

A very simple problem in geometry is measuring arclength (for more information on differential geometry, see [2]). To progress in our argument, we must define how it is measured. As stated in McLeary's paper, the length of a curve $\alpha : [t_0, t_1] \to U$ is given by the integral

$$\int_{t_0}^{t_1} ds = \int_{t_0}^{t_1} \sqrt{E(u(t), v(t))\left(\frac{du}{dt}\right)^2 + 2F(u(t), v(t))\frac{du}{dt}\frac{dv}{dt} + G(u(t), v(t))\left(\frac{dv}{dt}\right)^2}$$

where $(u(t), v(t)) = X(\alpha(t))$, and

$$E(u, v) = \frac{\partial X^{-1}}{\partial u} \cdot \frac{\partial X^{-1}}{\partial u}, \quad F(u, v) = \frac{\partial X^{-1}}{\partial u} \cdot \frac{\partial X^{-1}}{\partial v}, \quad G(u, v) = \frac{\partial X^{-1}}{\partial v} \cdot \frac{\partial X^{-1}}{\partial v}$$

Here, "$\cdot$" denotes the dot product, and the *metric*, ds, is given by

$$ds^2 = E(u, v)du^2 + 2F(u, v)\, du\, dv + G(u, v)dv^2$$

More information on metrics is available in [2], Chapter 8.

2.2 Central Projection

To derive the geometry on which to base our argument, we consider the *central projection*. The central projection is a mapping X from the lower hemisphere $U = \{(x, y, z) \in S_R^2 | z < 0\}$ to the plane $T = \{(x, y, -R) \in \mathbb{R}^3\}$, which we will equate with $\mathbb{R}^2$. Consider a non-horizontal line from the center of S_R^2 through some point in U. Then X maps that point to the intersection of the line and T.

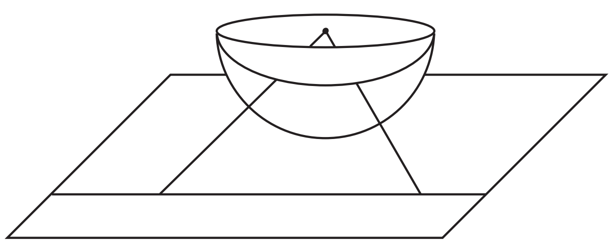

Figure 2: Central Projection

An interesting and important property of the central projection is that, for some surfaces, it maps the *geodesics* to straight lines, where a geodesic is the curve of shortest length between two points on a given surface. In the case of the sphere, these are the great circles.

Example 1. *Here we derive the coordinates given by the central projection. Let the points on the lower hemisphere of radius R be given by (a, b), where $a \in (-\pi, \pi)$ is the longitude and $b \in (-\pi/2, 0)$ is the latitude. Consider the right triangle compsed of the line from the origin to the center of the sphere, the line from the (a, b) to $X(a, b)$, and the line from $X(a, b)$ to the origin. Then the horizontal distance from (a, b) to the center of the sphere is given by $R\cos(b)$, and the vertical distance is $-R\sin(b)$.*

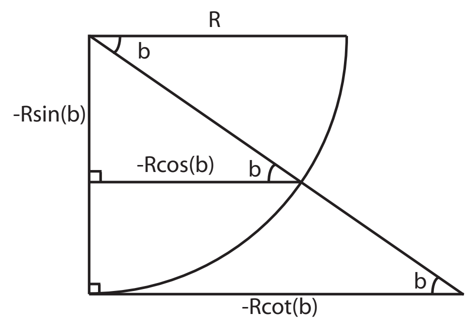

Figure 3: Similar Triangles

By similar triangles, the distance from the origin to $X(a, b)$ is $-R\cot(b)$. Furthermore, since a refers to a rotation about the vertical axis of the sphere, the image X must be correspondingly rotated. Thus

$$X(a, b) = (-R\cos(a)\cot(b), -R\sin(a)\cot(b))$$

The metric corresponding to this mapping is

$$ds^2 = R^2 \frac{(R^2 + v^2)du^2 - 2uv\, du\, dv + (R^2 + u^2)dv^2}{R^2 + u^2 + v^2}$$

(for a derivation of this fact, see [2], page 219). Letting $q = 1/R^2$, the curvature of a surface, we find

$$ds^2 = \frac{(1 + qv^2)du^2 - 2quv\, du\, dv + (1 + qu^2)dv^2}{1 + qu^2 + qv^2} \tag{1}$$

When $q > 0$, this is simply the geometry of the sphere with radius $1/\sqrt{q}$. When $q = 0$, we simply have $ds^2 = du^2 + dv^2$, the metric for plane geometry. But when $q < 0$, we have the geometry for a surface with *negative* curvature! With this in mind, we make a new definition.

Definition 2.2. *The* q-plane *is the subset $\mathbb{D}_q$ of $\mathbb{R}^2$ given by*

$$\mathbb{D}_q = \{(u, v)|1 + qu^2 + qv^2 > 0\}$$

with the metric given by (1).

When $q \geq 0$, this is the entire plane, as we would expect from our geometrical definition of the central projection. However, when $q < 0$, it is the open disc of radius $1/\sqrt{-q}$ centered at the origin. Now, since the central projection carries geodesics on our surface to straight lines in $\mathbb{D}_q$, we can learn about the geometry of a surface by examining straight lines in its image, $\mathbb{D}_q$.

2.3 Sides of a Triangle

We choose to consider a right triangle centered at the origin, and use ds to examine its sidelengths. We take one leg along the u-axis, one leg parallel to the v-axis, and the hypotenuse connecting the ends of those two legs.

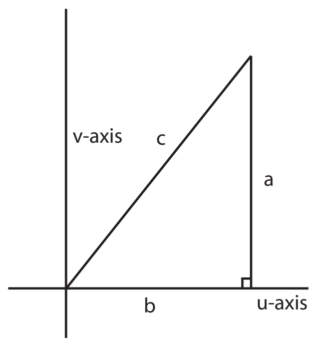

Figure 4: Lengths of Legs of a Right Triangle

First we find the length of side b. Parametrize b as $\beta(t) = (t, 0) = (u(t), v(t))$ for $0 \leq t \leq x$. Then $d\beta/dt = (1, 0) = (du(t), dv(t))$, $ds = (1 + qt^2)^{-1}dt$, and so since both v and dv are zero, we have

$$b = \int_0^x \frac{dt}{1 + qt^2}$$

For side a, let $\alpha(t) = (x, t)$ with $0 \leq t \leq y$. Then $d\alpha/dt = (0, 1)$, and, making use of (1), $ds = \sqrt{1 + qx^2}dt/(1 + qx^2 + qt^2)$. So integrating, we get

$$a = \int_0^y \frac{\sqrt{1 + qx^2}}{1 + qx^2 + qt^2}dt = \int_0^y \frac{d(t/\sqrt{1 + qx^2})}{1 + q(t/\sqrt{1 + qx^2})^2} = \int_0^{y/\sqrt{1+qx^2}} \frac{dt}{1 + qt^2}$$

For side c, we take $\gamma(t) = (tx, ty)$ for $0 \leq t \leq 1$. Then $d\gamma/dt = (x, y)$ and

$$c = \int_0^1 \sqrt{\frac{(1 + qt^2y^2)x^2 - 2qt^2x^2y^2 + (1 + qt^2x^2)y^2}{(1 + qt^2x^2 + qt^2y^2)^2}}dt$$

$$= \int_0^1 \frac{\sqrt{x^2 + y^2}dt}{1 + q(x^2 + y^2)t^2} = \int_0^1 \frac{d(\sqrt{x^2 + y^2}t)}{1 + q(\sqrt{x^2 + y^2}t)^2} = \int_0^{\sqrt{x^2+y^2}} \frac{dt}{1 + qt^2}$$

2.4 Trigonometric Functions

Notice that in the equation for the length of each side, the integrand is the same. In light of this fact, we create a new function.

Definition 2.3. *For $s \in \mathbb{R}$, define $\tau_q(s)$ implicitly by*

$$s = \int_0^{\tau_q(s)} \frac{dt}{1 + qt^2}$$

Example 2. *Let $q = 1$. Then $s = \int_0^{\tau_1(s)} \frac{dt}{1+t^2} = \arctan \tau_1(s)$. So*

$$\tau_1(s) = \tan(s) \tag{2}$$

Example 3. *Let $q = -1$. Then $s = \int_0^{\tau_{-1}} \frac{dt}{1-t^2}$. A simple application of partial fractions gives $s = \frac{1}{2}(\log(1+\tau_{-1}(s)) - \log(1-\tau_{-1}(s)))$. Then, using the division identity for logarithms and solving for $\tau_{-1}(s)$, we find*

$$\tau_{-1}(s) = \frac{e^{2s} - 1}{e^{2s} + 1} = \tanh s \tag{3}$$

These examples suggest that we consider $\tau_q(s)$ as the tangent function in $\mathbb{D}_q$. To determine the other trigonometric functions, we observe that

$$\frac{d\tau_q(s)}{ds} = 1 + q\tau_q^2(s)$$

Recalling that $\frac{d}{ds}\tan(s) = \sec^2(s) = 1 + \tan^2(s)$, we have a motivation for defining the following functions to act as sine and cosine.

Definition 2.4. *For $s \in \mathbb{R}$, define*

$$\sigma_q(s) = \frac{\tau_q(s)}{\sqrt{1 + q\tau_q^2(s)}}, \qquad \xi_q(s) = \frac{1}{\sqrt{1 + q\tau_q^2(s)}} \tag{4}$$

3 Infinite Series

We now derive some facts about our new trigonometric functions.

Example 4. *To find the Taylor Series of $\xi_q(s)$ and $\sigma_q(s)$, we first use the quotient rule for differentiation on $\xi_q(s)$. Combined with (5), this gives*

$$\frac{d\xi(s)}{ds} = \frac{\frac{-2q\tau_q(s)(1+q\tau_q^2(s))}{2\sqrt{1+q\tau_q^s}}}{1 + q\tau_q^2(s)} = -q\tau_q(s)\xi_q(s) = -q\sigma_q(s) \tag{5}$$

after cancellation. Next, we differentiate $\sigma_q(s)$ using the product rule

$$\frac{d\sigma_q(s)}{ds} = (1 + q\tau_q^2(s))\xi_q(s) - q\tau_q(s)\sigma q(s) = \xi_q(s) \tag{6}$$

From our definitions, we see that $\xi_q(0) = 1$ *and* $\sigma_q(s) = 0$*, so for* $s \in \mathbb{R}$*, we have the following Taylor Series*

$$\xi_q(s) = 1 - \frac{qs^2}{2!} + \frac{q^2 s^4}{4!} - \frac{q^3 s^6}{6!} + \cdots \tag{7}$$

$$\sigma_q(s) = s - \frac{qs^3}{3!} + \frac{q^2 s^5}{5!} - \frac{q^3 s^7}{7!} + \cdots \tag{8}$$

Note the similarities between these Taylor Series and those of the trigonometric functions. When $q = 1$*, the above series reduce to* $\cos(s)$ *and* $\sin(s)$*. When* $q = -1$*, they reduce to* $\cosh(s)$ *and* $\sinh(s)$*.*

To gain further insight into the trigonometric functions, we examine them in terms of a new function.

Definition 3.1. *The power series* $F_c(s)$ *is defined for* $c > 0$ *and* $s \in \mathbb{R}$ *by*

$$F_c(s) = 1 - \frac{s^2}{c} + \frac{s^4}{2!c(c+1)} - \frac{s^6}{3!c(c+1)(c+2)} + \cdots$$

In the context of this new function, we can rewrite (4) as $\sigma_q(s) = sF_{3/2}(s\sqrt{q}/2)$ and $\xi_q(s) = F_{1/2}(s\sqrt{q}/2)$. Notice here that if $q < 0$, we are actually giving a *complex* argument to $F_c(s)$, even though we defined it originally for only $s \in \mathbb{R}$. However, going back to the definition, we see that $F_c(s)$ is still real for $s \in \mathbb{C}$ provided that either $Re(s)$ or $Im(s)$ is zero. This ensures that s^{2n}, the only terms involving s in our definition of $F_c(s)$, are real.

4 Infinite Continued Fractions

We now consider our functions as infinite continued fractions to gain further insight.

Lemma 4.1. *The series* $F_c(s)$ *satisfies the identity*

$$F_{c+1}(s) - F_c(s) = \frac{s^2}{c(c+1)} F_{c+2}(s)$$

Proof. Since $F_c(s)$ defines an absolutely convergent series, we can rearrange terms as we please. Let $F_c(s, n)$ be the nth term of $F_c(s)$, with n starting at 0. We then have the formula

$$F_c(s, n) = \frac{(-s^2)^n}{n!c(c+1)\cdots(c+n-1)}$$

for $n > 0$. Now consider the following difference

$$\begin{aligned}
F_{c+1}(s,n) - F_c(s,n) &= \frac{(-s^2)^n}{n!(c+1)(c+2)\cdots(c+n)} \\
&\quad - \frac{(-s^2)^n}{n!c(c+1)\cdots(c+n-1)} \\
&= \frac{s^2}{c(c+1)} \cdot \frac{(-s^2)^{n-1}}{(n-1)!(c+2)(c+3)\cdots(c+n)} \\
&= \frac{s^2}{c(c+1)} F_{c+2}(s, n-1)
\end{aligned}$$

with equality between the first and second expressions on the right side from the fact that $\frac{1}{n!}(\frac{1}{c+n} - \frac{1}{c}) = \frac{-1}{(n-1)!c(c+n)}$. So since the constant terms of $F_{c+1}(s)$ and $F_c(s)$ cancel, we have the desired result. □

From this lemma, we see that

$$\frac{F_c(s)}{F_{c+1}(s)} = 1 - \frac{s^2}{c(c+1)} \frac{F_{c+2}(s)}{F_{c+1}(s)}$$

Proceeding recursively, we get the infinite fractional expansion

$$\frac{F_c(s)}{F_{c+1}(s)} = 1 - \cfrac{s^2/c(c+1)}{1 - \cfrac{s^2/(c+1)(c+2)}{1 - \cfrac{s^2/(c+2)(c+3)}{1 - \cdots}}}$$

Since we have a formula for the function $\tau_q(s)$ in terms of the functions $\xi_q(s)$ and $\sigma_q(s)$, and in turn formulas for those in terms of $F_c(s)$, we can write them as infinite continued fractions.

$$\frac{\tau_q(s)}{s} = \frac{F_{3/2}(\sqrt{q}s/2)}{F_{1/2}(\sqrt{q}s/2)} = \cfrac{1}{1 - \cfrac{(qs^2/4)(1/2)(3/2)}{1 - \cfrac{(qs^2/4)(3/2)(5/2)}{1 - \cfrac{(qs^2/4)(5/2)(7/2)}{1 - \cdots}}}}$$

In order to further our understanding of $\tau_q(s)/s$, we must examine infinite continued fractions in a more general context. First, we consider their convergence.

4.1 Convergents

What follows includes arguments from and based on [1], pages 491-514, which contains a wealth of information on infinite continued fractions.

Definition 4.2. *For the remainder of this paper, we consider infinite continued fractions as follows. Let* $\{a_k\}_{k=1}^{\infty}$ *and* $\{b_k\}_{k=1}^{\infty}$ *be two sequences of positive integers. When we speak of convergence of infinite continued fractions, we are considering the convergence of*

$$\cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \cdots}}} \tag{9}$$

Definition 4.3. *Let* $p_0 = 1$, $p_1 = 0$, $q_1 = 1$, *and* $q_2 = a_1$. *Then define* p_n *and* q_n *recursively by*

$$p_n = a_{n-1}p_{n-1} - b_{n-1}p_{n-2}$$

$$q_n = a_{n-1}q_{n-1} - b_{n-1}q_{n-2}$$

Then we say the convergents of a infinite continued fraction are the ratios p_n/q_n.

With this definition, we see that

$$\frac{p_1}{q_1} = 0, \quad \frac{p_2}{q_2} = \frac{b_1}{a_1}, \quad \frac{p_3}{q_3} = \cfrac{b_1}{a_1 - \cfrac{b_2}{a_2}}, \quad \frac{p_4}{q_4} = \cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cfrac{b_3}{a_3}}}, \quad etc. \tag{10}$$

That is, when we are testing infinite continued fractions of the form (9) for convergence, we say that the value of the infinite continued fraction is $lim_{n\to\infty} p_n/q_n$. We now consider possible convergence and divergence results of infinite continued fractions.

Example 5. *The infinite continued fraction*

$$\cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1 - \cdots}}}$$

does not converge. Depending on the values of n, p_n/q_n *alternates between* 0, 1, *and* ∞. *This is an example of oscillating divergence.*

Example 6. *The infinite continued fraction*

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = c, \qquad where \;\; c - 1 = 1/c \tag{11}$$

converges. Solving the quadratic equation in c, we find that the solution is the Golden Ratio, $c = \frac{1+\sqrt{5}}{2}$. *Notice that, when we consider the convergents, the* *infinite continued* $\dfrac{1}{\frac{1+\sqrt{5}}{2}-c}$ *diverges to infinity, a different type of divergence than we saw in the previous example.*

Lemma 4.4. *If $a_n \geq b_n + 1$ for all n, the convergents form a non-negative, increasing sequence.*

Proof. The first case is simple: $p_1/q_1 = 0 < b_1/a_1 = p_2/q_2$. For larger n, we use induction. Since the sequences for a and b both consist of positive integers, and our hypothesis gives that $a_n \geq b_n + 1 > b_n$ we know $a_{n-1} > a_{n-1} - b_n/a_n$. So $0 < \dfrac{b_{n-1}}{a_{n-1}} < \dfrac{b_{n-1}}{a_{n-1} - b_n/a_n}$. Furthermore, from our hypothesis relating a_n and b_n, we know that

$$1 > b_n/a_n, \text{ so } a_{n-1} - b_n/a_n > a_{n-1} - 1 \geq b_{n-1}, \text{ so } \frac{b_{n-1}}{a_{n-1} - b_n/a_n} < 1$$

Thus, because a_{n-2} is a positive integer and therefore greater than 1,

$$a_{n-2} - \frac{b_{n-1}}{a_{n-1} - b_n/a_n} > 0 \tag{12}$$

for all n greater than 3. This is important because it means that since the left side of (12) is nonnegative, we do not have to worry about the convergents changing signs. Furthermore, since the left side is nonzero, the convergents are always defined. Now, suppose that p_n/q_n is positive. Then

$$\frac{p_n}{q_n} = \cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cdots \cfrac{b_{n-2}}{a_{n-2} - \cfrac{b_{n-1}}{a_{n-1}}}}} < \cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cdots \cfrac{b_{n-2}}{a_{n-2} - \cfrac{b_{n-1}}{a_{n-1} - \cfrac{a_n}{b_n}}}}} = \frac{p_{n+1}}{q_{n+1}}$$

□

We next consider another property of convergents.

Lemma 4.5. *If $a_n \geq b_n + 1$ for all n, then $q_n \geq 1 + b_1 + b_1 b_2 + \cdots + b_1 b_2 \cdots b_{n-1}$.*

Proof. First, we observe that

$$\begin{aligned}
q_n - q_{n-1} &= a_{n-1}q_{n-1} - b_{n-1}q_{n-2} - a_{n-2}q_{n-2} + b_{n-2}q_{n-3} \\
&= a_{n-1}q_{n-1} - b_{n-1}q_{n-2} - a_{n-2}q_{n-2} + (a_{n-2}q_{n-2} - q_{n-1}) \\
&= b_{n-1}\left(\frac{a_{n-1} - 1}{b_{n-1}} q_{n-1} - q_{n-2}\right) \\
&\geq b_{n-1}(q_{n-1} - q_{n-2})
\end{aligned}$$

because $a_{n-1} - 1 \geq b_{n-1}$. Using this inequality recursively, we see that

$$q_n - q_{n-1} \geq b_1 b_2 \cdots b_{n-1}$$

Now, making repeated use of this result, we get

$$\begin{aligned} q_n &= (q_n - q_{n-1}) + (q_{n-1} - q_{n-2}) + \cdots (q_2 - q_1) + q_1 \\ &\geq 1 + b_1 + b_1 b_2 + \cdots + b_1 b_2 \cdots b_{n-1} \end{aligned}$$

□

To get a more general result regarding infinite continued fractions, we need one more preliminary result.

Lemma 4.6. *If $a_n \geq b_n + 1$ for all n, then $q_n - p_n \geq q_{n-1} - p_{n-1} \geq \cdots \geq q_2 - p_2 \geq 1$.*

Proof. In the case $n = 2$, $q_2 - p_2 = a_1 - b_1 \geq 1$. When $n = 3$, we have

$$\begin{aligned} q_3 - p_3 &= a_1 a_2 - b_2 - b_1 a_2 \\ &\geq (a_1 - b_1)(b_2 + 1) - b_2 \\ &= (a_1 - b_1) + b_2(a_1 - b_1 - 1) \\ &= (q_2 - p_2) + b_2((q_2 - p_2) - (q_1 - p_1)) \end{aligned}$$

This form of $q_3 - p_3$ can serve as the base case for an inductive argument. We notice that

$$\begin{aligned} & (q_{n-1} - p_{n-1}) + b_{n-1}((q_{n-1} - p_{n-1}) - (q_{n-2} - p_{n-2})) \\ = \ & (q_{n-1} - p_{n-1})(b_{n-1} + 1) - b_{n-1}(q_{n-2} - p_{n-2}) \\ \leq \ & a_{n-1}(q_{n-1} - p_{n-1}) - b_{n-1}(q_{n-2} - p_{n-2}) \\ = \ & q_n - p_n \end{aligned}$$

Thus, if $q_{n-1} - p_{n-1} \geq q_{n-2} - p_{n-2} \geq 1$, then $q_n - p_n \geq q_{n-1} - p_{n-1}$. □

4.2 Convergence and Irrationality

Theorem 4.7. *Suppose $a_n \geq b_n + 1$ for all values of n, with strict inequality holding at least once. Then the infinite continued fraction of the form (9) converges to a positive limit $F < 1$.*

Proof. By Lemma 3.4, the convergents form a positive, increasing sequence, so we cannot have the oscillation we saw in Example 6. Then because $q_n - p_n \geq 1$, $p_n/q_n \leq 1 - 1/q_n$. So since $q_n > 1$, we have $0 < F < 1$. □

Theorem 4.8. *The infinite continued fraction of the form (9) converges to an irrational limit if there exists N such that $a_n \geq b_n + 1$ for all $n > N$, with $a_n > b_n + 1$ occurring infinitely often.*

Proof. It suffices to show that the infinite continued fraction

$$\cfrac{b_N}{a_N - \cfrac{b_{N+1}}{a_{N+1} - \cfrac{b_{N+2}}{a_{N+2} - \cdots}}} \tag{13}$$

is rational (that the theorem follows is left to the reader). Then (13) converges, by Theorem 4.7, to a positive value less than 1. Now, assume that (13) is rational. Then we can write it as λ_2/λ_1, with λ_1 and λ_2 both positive integers, and $\lambda_1 > \lambda_2$. So let

$$\rho_1 = \cfrac{b_{N+1}}{a_{N+1} - \cfrac{b_{N+2}}{a_{N+2} - \cdots}}$$

Since we know $a_n > b_n$ for infinitely many n, ρ_1 must be positive and less than 1. Now, $\lambda_2/\lambda_1 = b_N/(a_N - \rho_1)$, so $\rho_1 = \lambda_3/\lambda_2$ where $0 < \lambda_3 = a_2\lambda_2 - b_2\lambda_1 < 1$ since $0 < \rho_1 < 1$. Then if we let

$$\rho_2 = \cfrac{b_{N+2}}{a_{N+2} - \cfrac{b_{N+3}}{a_{N+3} - \cdots}}$$

we find that $\rho_2 = \lambda_4/\lambda_3$, with $0 < \lambda_4 < \lambda_3$. Proceeding inductively, we see that our assumption that (13) is rational requires that there exist an infinite sequence of integers $\{\lambda_k\}_{k=1}^{\infty}$ such that $\lambda_k > \lambda_{k+1}$ for all k. However, λ_1 is finite, so we have a contradiction. Thus (13) is irrational. □

Theorem 4.9. *If s and q are rational, then $\tau_q(s)$ is irrational.*

Proof. Since q and s are rational, we can write $qs^2 = m/n$. Now suppose $\tau_q(s)/s$ were rational, then we could say $\tau_q(s)/s = a/b$. Then

$$\frac{a}{b} = \cfrac{1}{1 - \cfrac{m/n}{3 - \cfrac{m/n}{5 - \cfrac{m/n}{7 - \cdots}}}} = \cfrac{n}{n - \cfrac{mn}{3n - \cfrac{mn}{5n - \cfrac{mn}{7n - \cdots}}}}.$$

Thus, since mn is constant, the hypothesis of the previous theorem are satisfied. So $\tau_q(s)$ cannot be rational. □

We our now ready to give our final result.

Corollary 4.10. *π and e are irrational.*

Proof. To show π is irrational, let $q = 1$ and $s = \pi/4$. Then since $\tau_1(x) = \tan(x)$, we see $\tau_q(s)/s = \tan(\pi/4)/(\pi/4) = 4/\pi$. So Theorem 4.9 is contradicted if π is rational. To show e is irrational, let $q = -1$ and $s = 1$. Because $\tau_{-1}(x) = \tanh(x)$, $\tau_q(s)/s = (e - e^{-1})/(e + e^{-1})$. Thus, e must be irrational to maintain consistency with Theorem 4.9. □

5 Conclusion

McCleary's paper is interesting because it sets the proofs of the irrationalities of π and e in a general context—a context that one might expect to have little to do numberical properties. It is important to be able to unite different parts of mathematics in this way, as it allows us to use theorems for one topic and apply them to another. Thus, we gain new methods that may allow us to solve problems which previously held an elusive solution.

References

[1] G. Chrystal. *Algebra, An Elementary Text-Book*, volume 2. Chelsea Publishing Company, 1959.

[2] John McCleary. *Geometry from a Differentiable Viewpoint*. Cambridge University Press, 1994.

[3] John McCleary. Trigonometries. *The American Mathematical Monthly*, 109(7):623–638, August–September 2002.