Analytic Continuation of Zeta

Note Title 6/4/2008

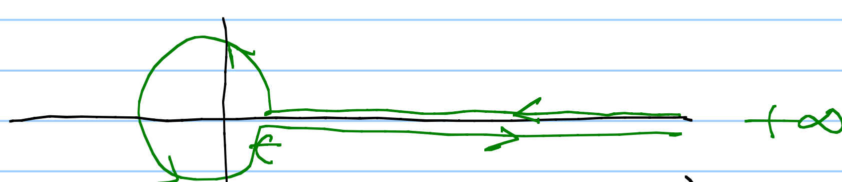
For $\mathrm{Re}(z) > 1$,

$$\zeta(z) = \sum \frac{1}{n^z} = \prod_{p \in P} \left(1 - \frac{1}{p^z}\right)^{-1} = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt$$

Each expression is analytic, converges for $\mathrm{Re}(z) > 1$, and does not converge when $z = 1$.

The following theorem due to Riemann provides the analytic continuation of ζ to a function analytic on $\mathbb{C} - \{1\}$ with simple pole at $z = 1$.

Let Γ_ϵ be the contour that starts at $+\infty$ on the positive real axis comes in to the point $\epsilon > 0$, circles the origin in a counterclockwise direction on $|z| = \epsilon$ and returns to $+\infty$ on the positive real axis.



Let $$f(z) = \int_{\Gamma_\epsilon} \frac{w^{z-1}}{e^w - 1}\, dw = \int_{+\infty}^{(0+)} \frac{w^{z-1}}{e^w - 1}\, dw$$

Theorem: $f(z)$ is an entire function.

$$\zeta(z) = \frac{f(z)}{\Gamma(z)(e^{2\pi i z} - 1)} \quad \text{for } \mathrm{Re}(z) > 1$$

Since $\Gamma(z)$ has simple poles at $0, -1, -2, \dots$

and $e^{2\pi i z} - 1$ has simple zeros at $n \in \mathbb{Z}$,

$\Gamma(z) \cdot (e^{2\pi i z} - 1)$ is analytic, and never 0 for

$z \neq 1, 2, \dots$. In other words the expression

$\dfrac{f(z)}{\Gamma(z)(e^{2\pi i z}-1)}$ is analytic everywhere except possibly for

simple poles at $1, 2, \dots$. Since it equals $\zeta(z)$

for $\mathrm{Re}(z) > 1$ it can only have a simple pole at $z=1$

(which it does and the residue is 1; not proved here).

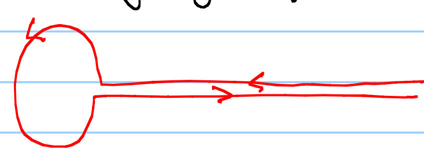
Now to a proof of the theorem and an

explanation of the meaning of $\displaystyle\int_{\Gamma_\epsilon} \frac{w^{z-1}}{e^w - 1}\, dw$.

In this integral, either w is real, $w \geq \epsilon$ or $|w| = \epsilon$.

Redraw the contour separating the ray coming in from

$+\infty$ to ϵ from the ray going out from ϵ to $+\infty$.



On the top edge we define $\log w = \log|w| = \log t$.

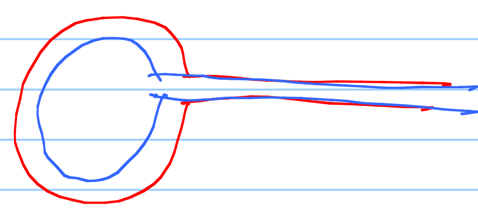
On the bottom edge $\log w = \log t + 2\pi i$. On the

curve $|w| = \epsilon$, $\log w$ varies continuously from

$\log\epsilon$ to $\log\epsilon + 2\pi i$. Using this definition the integral is a sum of three terms

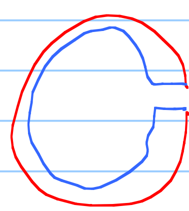
$$\int_{\infty}^{\epsilon} \frac{t^{z-1}}{e^t-1}\,dt + e^{2\pi i(z-1)} \int_{\epsilon}^{\infty} \frac{t^{z-1}\,dt}{e^t-1} + \int_{|w|=\epsilon} \frac{w^{z-1}}{e^w-1}\,dw$$

First we notice that this sum is independent of ϵ. This can be seen by looking at the difference between two such integrals and separating the portions along the x-axis by a little bit.



Difference

=



The integral over this simple closed curve of the function $\frac{w^{z-1}}{e^w-1}$, analytic in w (for fixed z) in the nearly annular region, is 0. Hence the integral is independent of ϵ.

Let's compute the limit as $\epsilon \to 0$.

the two ray integrals combine to give

$$\left(e^{2\pi i z}-1\right)\int_0^\infty \frac{t^{z-1}}{e^t-1}\,dt$$

Next we estimate $\displaystyle\int_{|w|=\epsilon} \frac{w^{z-1}}{e^w-1}\,dw$

$$\left|e^w-1\right| = \left|w\left(1+\frac{w}{2}+\frac{w^3}{3!}+\cdots\right)\right| > \frac{|w|}{2} \quad \text{for small } w.$$

$$w^{z-1} = e^{(z-1)(\log|w|+i\arg w)}$$

$$\left|w^{z-1}\right| = e^{\mathrm{Re}(z-1)\log|w| + (\arg w)\,\mathrm{Im}(z-1)}$$

$$\leq |w|^{\mathrm{Re}(z-1)}\cdot e^{2\pi\,\mathrm{Im}(z-1)} = M|w|^{\mathrm{Re}(z)-1}$$

where M depends on z, but z is fixed.

Suppose $\mathrm{Re}(z)=s>1$. Then

$$\left|\frac{w^{z-1}}{e^w-1}\right| \leq \frac{2}{|w|}\cdot M|w|^{s-1} = \frac{M}{2}|w|^{s-2} = \frac{M}{2}\epsilon^{s-2}$$

Hence $\displaystyle\left|\int\frac{w^{z-1}}{e^w-1}\,dw\right| \leq (2\pi\epsilon)\frac{M}{2}\epsilon^{s-2} = \pi M\epsilon^{s-1}$

and since $s>1$, this goes to 0 as $\epsilon\to 0$.

So we have an entire function $f(z)$,

which for $z=s>1$ equals

$$\left(e^{2\pi i s}-1\right)\int_0^\infty \frac{t^{s-1}}{e^s-1}\,dt.$$

Hence $\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{s}-1}\,dt$

$$= \left[\frac{1}{\Gamma(s)(e^{2\pi i s}-1)}\right] f(s)$$

The function $$\frac{f(z)}{\Gamma(z)(e^{2\pi i z}-1)}$$

is analytic everywhere, except possibly at $z = 1, 2, 3, \cdots \in \mathbb{Z}$ (the positive integers).

This gives an analytic continuation of $\zeta(z)$ to all of $\mathbb{C} - \{1, 2, \cdots\}$. But since we already know $\zeta(z)$ is analytic for $\mathrm{Re}(z) > 1$, the only possible singular point is at $z = 1$. It can be proved that 1 is a simple pole of $\zeta(z)$ and the residue is $+1$.