Uniqueness for the Dirichlet Problem on $\mathbb{D}$

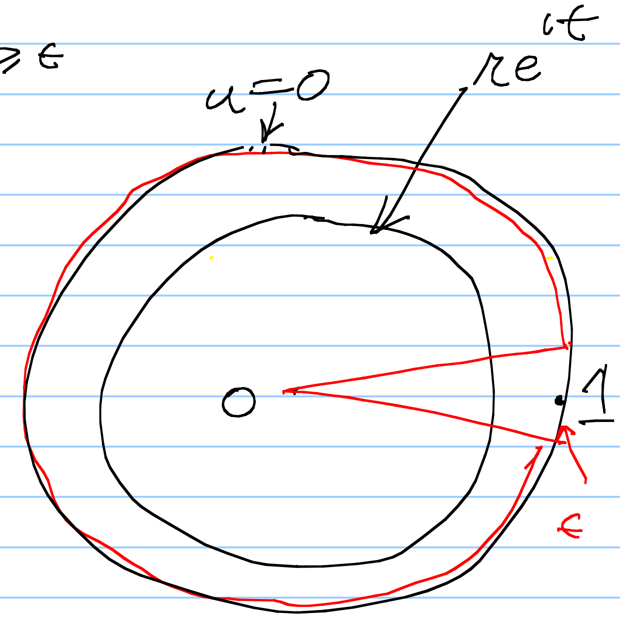
Note Title 5/14/2008

Let $u(z)$ be harmonic in $\mathbb{D}$ and suppose $\lim_{z \to \zeta} u(z) = 0$ for all $\zeta \in \partial\mathbb{D} - \{\zeta_1, \dots, \zeta_k\}$. Suppose also that $|u(z)| \leq M$. Then $u(z) = 0$ for all $z \in \mathbb{D}$.

Proof: (Owen Biesel). To simplify the argument, suppose $k=1$ and $\zeta_1 = 1$. By the mean value theorem $u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{it})\,dt$ if $0 \leq r < 1$.

$$u(0) = \int_0^{2\pi} u(re^{it})\,dt = \int_{|t|<\epsilon} + \int_{|t| \geq \epsilon} = \mathrm{I} + \mathrm{II}$$

$$\mathrm{I} = \left| \int_{|t|<\epsilon} u(re^{it})\,dt \right| \leq M\epsilon.$$



$u(re^{it})$ is continuous on region bounded by the red arc and red wedge.

$u(re^{it}) \to 0$ as $r \to 1$, uniformly in t, on this region since u is uniformly continuous on this compact set.

Hence $\mathrm{II} = \int_{|t| \geq \epsilon} u(re^{it})\,dt \to 0$ as $r \to 1$.

This proves that $u(0) = 0$.

Now let $\alpha \in \mathbb{D}$. Let $w(z) = u\left(\frac{\alpha - z}{1 - \bar{\alpha} z}\right)$.
Then w is harmonic and $w(z) \to 0$ as $z \to \partial\mathbb{D}$, except possibly at one point. By the previous argument, $w(0) = 0$. But $w(0) = u(\alpha)$, so $u(\alpha) = 0$. This is true for any $\alpha \in \mathbb{D}$, hence $u \equiv 0$.

See if you can prove the following result.
Let u be harmonic in $\mathbb{D}$. Suppose $\limsup_{z \to \zeta} u(z) \le 0$ for all points $\zeta \in \mathbb{D} - \{\zeta_1, \ldots, \zeta_k\}$ and also $u(z) \le M$.
Prove that $u(z) \le 0$ for all $z \in \mathbb{D}$.