Euler's Product Formula

Note Title 5/21/2008

Let s be a real number $s > 1$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{\left(1 - \frac{1}{p^s}\right)}, \text{ where } P \text{ is}$$

the set of all primes $P = \{2, 3, 5, 7 \cdots\}$

Proof: List the primes in a sequence $p_1 < p_2 < p_3 < \cdots$

$p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\ldots$. By $\prod_{p \in P} \frac{1}{\left(1 - \frac{1}{p^s}\right)}$ we

mean $\lim_{N \to \infty} \prod_{k=1}^{N} \frac{1}{\left(1 - \frac{1}{p_k^s}\right)}$, if it exists.

Each term in the product is the sum of a geometric series $1 + \frac{1}{p_k^s} + \frac{1}{(p_k^s)^2} + \cdots + \frac{1}{(p_k^s)^m} + \cdots$, which is an absolutely convergent sum of positive terms. Hence the finite product can be rearranged as:

$$1 + \frac{1}{p_1^s} + \frac{1}{p_2^s} + \cdots + \frac{1}{p_N^s} + \frac{1}{(p_1^2 p_2)^s} + \frac{1}{(p_1 p_2^2)^s} + \cdots$$

$$= \sum_{m = (m_1, \ldots, m_N)} \frac{1}{\left[p_1^{m_1} \cdots p_N^{m_N}\right]^s} = \sum \frac{1}{q^s}$$

where q is a product of powers of $p_1, \ldots, p_N$.

Now consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} - \prod_{k=1}^{N} \frac{1}{\left(1 - \frac{1}{p_k^s}\right)} = D$.

We have removed from $\sum \frac{1}{n^s}$ all terms $\frac{1}{q^s}$ where the only prime factors of q are $p_1, \dots, p_N$. The remaining terms are of the form $\frac{1}{q^s}$ where at least one prime factor of q is larger than p_N. In other words every remaining term is among the terms in the sum $\displaystyle\sum_{n \geq p_{N+1}} \frac{1}{n^s}$. Hence

$D \leq \displaystyle\sum_{n \geq p_{N+1}} \frac{1}{n^s} \longrightarrow 0$ as $N \to \infty$. We've proved

$$\text{(F)} \quad \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{\left(1 - \frac{1}{p^s}\right)} \quad \text{for } s > 1.$$

Now we already know $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ is analytic when $Re(s) > 1$. If we can prove that $\displaystyle\prod_{p \in P} \frac{1}{\left(1 - \frac{1}{p^s}\right)}$ is analytic when $Re(s) > 1$, the identity theorem will guarantee (F) for $Re(s) > 1$.

Each term $1-\frac{1}{p^s}$ is analytic in $s=\sigma+it$ when $\sigma>1$ and different from 0. $|p^s|=p^\sigma$ and $\sum_{p\in P}\frac{1}{p^\sigma}$ converges if $\sigma>1$. This implies that $\prod_{p\in P}\left(1-\frac{1}{p^s}\right)$ converges uniformly and absolutely on compact subsets of $\Re(s)>1$ to an analytic function. By Hurwitz's theorem $\prod\left(1-\frac{1}{p^s}\right)$ has no zeros so $\prod\left(\frac{1}{1-\frac{1}{p^s}}\right)$ is analytic when $\Re(s)>1$.