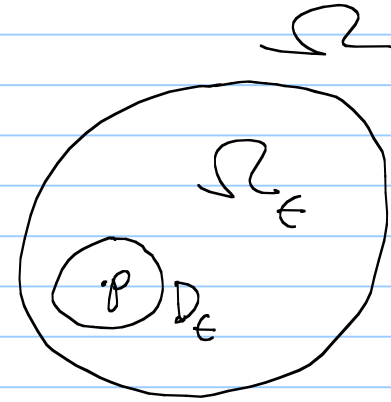
Poisson Integral Formula

Note Title 5/19/2008

Let u be harmonic on $\overline{\Omega}$, where Ω is open, connected, bounded, and $\partial\Omega$ is an analytic curve. Then if $p \in \Omega$ and $\phi = u$ on $\partial\Omega$.

$$u(p) = -\frac{1}{2\pi}\int_{\partial\Omega} \phi(s)\,\frac{\partial g_p(s)}{\partial n(s)}\,ds(s)$$



Proof: $\Omega_\epsilon = \Omega - \overline{D}_\epsilon$

where $D_\epsilon = \{z : |z-p| < \epsilon\}$

$$\int_{\partial\Omega_\epsilon}\left(g_p\frac{\partial u}{\partial n} - u\frac{\partial g_p}{\partial n}\right)ds = \int_{\Omega_\epsilon}\left(g_p\nabla^2 u - u\nabla^2 g_p\right)dA = 0$$

Since $g_p = 0$ on $\partial\Omega$, we get

$$(\mathrm{I}) \quad -\int_{\partial\Omega}\phi\,\frac{\partial g_p}{\partial n}\,ds = \int_{\partial D_\epsilon}\left(g_p\frac{\partial u}{\partial n} - u\frac{\partial g_p}{\partial n}\right)ds$$

Exactly as in the proof of symmetry of the Green's Function, the limit as $\epsilon \to 0$ of the right side of (I) is $2\pi u(p)$. This proves the result.