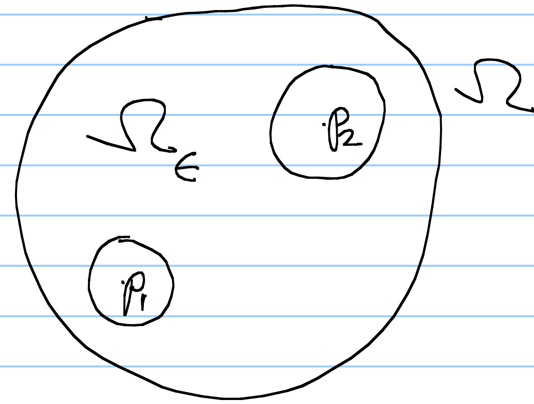
Symmetry of Green's Function



$$g_{p_1}(p_2) = g_{p_2}(p_1)$$

$$D_{1\epsilon} = \{z : |z - p_1| < \epsilon\}$$

$$D_{2\epsilon} = \{z : |z - p_2| < \epsilon\}$$

$$\Omega_\epsilon = \Omega - (\overline{D_{1\epsilon}} \cup \overline{D_{2\epsilon}})$$

$$\int_{\partial\Omega_\epsilon} \left(g_{p_1} \frac{\partial g_{p_2}}{\partial n} - g_{p_2} \frac{\partial g_{p_1}}{\partial n}\right) ds$$

$$= \int_{\Omega_\epsilon} \left(g_{p_1} \nabla^2 g_{p_2} - g_{p_2} \nabla^2 g_{p_1}\right) dA = 0. \text{ so}$$

$$(\mathrm{I}) \quad \int_{\partial D_{1\epsilon}} (\quad) + \int_{\partial D_{2\epsilon}} (\quad) = \int_{\partial\Omega} (\quad) = 0$$

since g_1 and g_2 are 0 on $\partial\Omega$.

On $D_{1\epsilon}$ $\left|\frac{\partial g_{p_2}}{\partial n}\right| = |\nabla g_{p_2} \cdot n| \le M_1$ and

$g_{p_1} = -\log|z - p_1| + h$. Hence

$$\left| \int_{\partial D_{1\epsilon}} g_{p_1} \frac{\partial g_{p_2}}{\partial n} ds \right| \le (|\log \epsilon| + M_2) M_1 2\pi\epsilon$$

$$\longrightarrow 0 \text{ as } \epsilon \to 0$$

$$-\int_{\partial D_{1\epsilon}} g_{p_2} \frac{\partial g_{p_1}}{\partial n} ds = -\int_0^{2\pi} g_{p_2}(p_1 + \epsilon e^{i\theta}) \left(-\frac{1}{\epsilon} + \nabla h \cdot n\right) \epsilon \, d\theta$$

since on $\partial D_{1\epsilon}$ $\dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial r}$ where $z = p_1 + re^{i\theta}$

and $\dfrac{\partial}{\partial n} \log|z - p_1| = \dfrac{\partial}{\partial r} \log r = \dfrac{1}{r}$ and

set $r = \epsilon$. as before $|\nabla h \cdot n| \leq M$ so

$$\lim_{\epsilon \to 0} \int_0^{2\pi} (g_{p_2})(\nabla h \cdot n) \epsilon \, d\theta = 0.$$

Finally

$$\lim_{\epsilon \to 0} - \int_{\partial D_1 \epsilon} g_{p_2} \frac{\partial g_{p_1}}{\partial n} \, ds = \lim_{\epsilon \to 0} \int_0^{2\pi} g_{p_2}(p_1 + \epsilon e^{i\theta}) \, d\theta$$

$$= 2\pi g_{p_2}(p_1)$$

We've computed the first term of (I). The second term is $-2\pi g_{p_1}(p_2)$

Since $2\pi g_2(p_1) - 2\pi g_{p_1}(p_2) = 0$,

we are done.