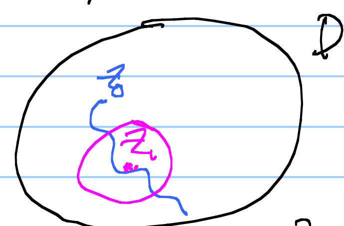
Mean Value Property

Note Title 5/15/2008

Let u be a continuous function on an open set Ω. Suppose that for every $z_0 \in \Omega$, there is an $r_0 > 0$ so that if $r \le r_0$ $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt$. Then u is harmonic.

Lemma: Let $\bar{D}$ be a closed disk in Ω. Let w have the mean value property at all points of $\bar{D}$. Then $\sup\{u(z) : z \in \bar{D}\} = \sup\{u(z) : z \in \partial D\}$.

Proof: Let $m = \sup\{u(z) : z \in \bar{D}\}$ and suppose $m = u(z_0)$, $z_0 \in D$. Let $\gamma(t)$, $t \in [0, a]$ be a curve in D from z_0 to a point on ∂D.



Let $t_1 = \sup\{t : u(\gamma(t)) = m\}$. If $\gamma(t_1) \in \partial D$ we are done, since by continuity $u(\gamma(t_1)) = m$. If $\gamma(t_1) = z_1 \in D$, let $|z - z_1| \le r_0$ be a small disk in which the mean value property holds. Then $0 = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{[u(z_1) - u(z_1 + re^{it})]}_{\ge 0}\,dt$ for all $r \le r_0$.

The only way this can happen is for $u(z_1) = u(z_1 + re^{it})$ for all $r \le r_0$, $0 \le t \le 2\pi$. So $u(z) = m$ on the disk $|z - z_1| \le r_0$. But then there is a $t > t_1$ with $u(\gamma(t)) = m$. This proves that $\sup\{u(z) : z \in D\} = \sup\{u(z) : z \in \partial D\}$

Now to prove the theorem: Let v be harmonic in D with $v = u$ on ∂D. Let $w = u - v$. Then w satisfies the MVP and $w = 0$ on ∂D. So $w(z) = 0$ for all $z \in D$. Hence $u = v$ is harmonic.