Schwarz's Formula

Note Title 5/9/2008

Suppose $f \in \mathcal{O}(\overline{\mathbb{D}})$. Then if $|z|<1$,

(0) $$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\left(\frac{\zeta+z}{\zeta-z}\right)dt + i v(0), \text{ where } \zeta = e^{it}$$

and $f = u + iv$.

Proof: $$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad d\zeta = i\zeta\,dt$$

(1) $$f(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(\zeta)\,\zeta}{\zeta - z}\,dt$$

If $z \neq 0$, $$0 = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta - 1/\bar{z}}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\bar{z}\,f(\zeta)}{\bar{z}\zeta - 1}\,d\zeta$$

and this is also true if $z=0$, so

$$0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{\bar{z}\,f(\zeta)}{\bar{z}\zeta - 1}\,\zeta\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{\bar{z}\,f(\zeta)}{\bar{z} - \bar{\zeta}}\,dt,$$

since $\zeta\cdot\bar{\zeta} = 1$. Conjugate and change signs

(2) $$0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{z\,\bar{f}(\zeta)}{\zeta - z}\,dt.$$

Add (1) and (2):

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\zeta - z}\left[\zeta f(\zeta) + z\bar{f}(\zeta)\right]dt$$

Next $\zeta f(\zeta) + z\bar{f}(\zeta)$

$$= (\zeta + z)u + (\zeta - z)(iv) \quad , \text{ so}$$

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\zeta+z}{\zeta-z}\right)u(\zeta)\,dt + \frac{i}{2\pi}\int_0^{2\pi} v(e^{it})\,dt$$

$$(3) \qquad f(z) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\zeta+z}{\zeta-z}\right)u(\zeta)\,dt + i\,v(0)$$

$$(4) \qquad u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\,\mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right)dt$$

$$(5) \qquad v(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\,\mathrm{Im}\left(\frac{\zeta+z}{\zeta-z}\right)dt + v(0)$$