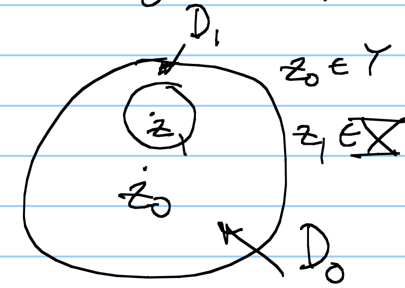
Identity Theorem for Harmonic Functions

Note Title 5/12/2008

Let u be harmonic on an open connected set Ω. If u is constant on an open subset of Ω, then u is constant.

Proof: Let $X = \{z \in \Omega : u \text{ is constant in a neighborhood of } z\}$. We assume $X \neq \emptyset$. Let $Y = \Omega - X$. Then Y is also open. If not, there is $z_0 \in Y$ and no neighborhood of z_0 is a subset of Y. Let D_0 be a disk centered at z_0, $D_0 \subset \Omega$. Then $D_0 \not\subset Y$ so there is a point $z_1 \in D_0 \cap X$, and then there is a disk D_1, $z_1 \in D_1$, $D_1 \subset D_0$ and u is constant on D_1. Since D_0 is a disk $u = \mathrm{Re}(f)$, f analytic on D_0, and since u is constant on D_1, f is constant on D_1 and hence constant on D_0. But then u is constant on D_0 and hence $D_0 \subset X$. But z_0 was in Y so this can't happen.



$\Omega = X \cup Y$, $X \cap Y = \emptyset$, X, Y open

$X \neq \emptyset \implies Y = \emptyset$. So u is locally constant. Now pick some point $a \in \Omega$. Let $c = u(a)$. Let $A = \{z : u(z) = c\}$

and let $B = \Omega - A$. A is open because u is locally constant. Also $A \neq \phi$, since $a \in A$.
B is open because u is continuous: if $u(z) \neq c$, then $u(z) \neq c$ in a neighborhood of z.

$\Omega = A \cup B$, $A \cap B = \phi$, A and B are open, $A \neq \phi$.

Hence $B = \phi$ and $u(z) = c$ for all $z \in \Omega$ (u is constant).