Note Title 4/26/2009

Triangle Inequality

Suppose $\left|\int g\right| = \int |g| \neq 0$

Let $\int g = c e^{i\phi}$, $c = \int |g| > 0$.

Let $h = e^{-i\phi} g$. Then $|h| = |g|$.

$\int h = e^{-i\phi} \int g = c > 0$, so $\int \mathrm{Re}(h) = c$,

But also $\int |h| = \int |g| = c$; hence

$\int(|h| - \mathrm{Re}\, h) = 0$. But $|h|(x) - \mathrm{Re}\, h(x) \geq 0$

all values of x, hence $\mathrm{Re}\; h(x) = |h|(x)$ for all x. This implies $\mathrm{Im}\, h(x) = 0$ for all x

and so $h(x) = |h(x)| \geq 0$ for all x.

In other words:

The only way to have equality in $\left|\int g\right| \leq \int |g|$ is for there to be a complex number ω, $|\omega| = 1$, so that $\omega \cdot g(x) \geq 0$ for all x. (After a rotation, all values of g are ≥ 0.