The Framework of Music Theory as Represented with Groups

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1 Introduction

In 2002, a music theorist by the name of Julian Hook published a paper in the *Journal of Music Theory* titled, "Uniform Triadic Transformations." In this paper, Hook generalized some existing music theoretical concepts and greatly improved their notation. Hook's UTTs formed a group with interesting algebraic properties.

This paper will first give the reader a review of all necessary group theory to understand the discussion of Hook's UTTs. Then it will review music theory (atonal theory in particular) and its evolution to the UTTs. Finally, it will discuss the UTTs themselves and conclude with some musical applications.

2 Basic Group Theory

Group theory is a branch of mathematics that studies groups. This algebraic structure forms the basis for abstract algebra, which studies other structures such as rings, fields, modules, vector spaces and algebras. These can all be classified as groups with addition operations and axioms.

This section provides a quick and basic review of group theory, which will serve as the basis for discussions in the group theoretical structure as applied to music theory. Readers interested in a more thorough discussion to group theory and abstract algebra may refer to [1] and [2].

2.1 What is a group?

A group is a set such that any two elements x and y can be combined via "multiplication" to form a unique product xy that also lies in the set. This multiplication is defined for every group and does not necessarily refer to the traditional meaning of "multiplication." We now state the formal definition of a group:

Definition 2.1. A **group** is a set G together with a multiplication on G which satisfies three axioms:

(a) The multiplication is associative, that is to say $(xy)z = x(yz)$ for any three (not necessarily distinct) elements in G.

(b) There is an element $e \in G$, called an identity element, such that $xe = x = ex$ for every $x \in G$

(c) Each element $x \in G$ has an inverse x^{-1} which belongs to the set G and satisfies $x^{-1}x = e = xx^{-1}$

Two important properties follow easily from the definition of a group. The proof of these properties is left to the reader.

Theorem 2.2. *Every group G satisfies the following properties:*

(a) The identity element e of G is unique

(b) For all $x \in G$, the inverse x^{-1} is unique

Note how commutivity of the multiplication is not required within a group. Therefore, we define an abelian group as follows:

Definition 2.3. A group G is **abelian** if its multiplication is commutative. That is, $xy = yx$ for any two elements in G.

To better illustrate the concept of a group, we now give some examples.

Example 2.4. The reals excluding 0 $\mathbb{R}\backslash\{0\}$ under multiplication:

- The group is closed under multiplication: for all $x, y \in \mathbb{R}$, $x \cdot y \in \mathbb{R}$.
- The multiplication is associative: for all $x, y, z \in \mathbb{R}$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- The identity is 1: for all $x \in \mathbb{R}$, $1 \cdot x = x = x \cdot 1$.
- The inverse of x is $\frac{1}{x}$: $\frac{1}{x} \cdot x = 1 = x \cdot \frac{1}{x}$.

This group is abelian since $x, y \in \mathbb{R}$, $x \cdot y = y \cdot x$. Note how we must exclude 0 for this to be a group since there exists no inverse for 0. That is, there does not exists some $x \in \mathbb{R}$ such that $x \cdot 0 = 1$

Example 2.5. The integers $\mathbb{Z}$ (mod 12), which we will denote as $\mathbb{Z}_{12}$, under addition (mod 12). (*Note:* in abstract algebra, $\mathbb{Z}$ (mod 12) is generally notated as $\mathbb{Z}/(12)$ and $\mathbb{Z}_{12}$ refers to another algebraic structure. However, in music theory, only $\mathbb{Z}$ (mod 12) is of significance and we will use this more concise notation.)

- The group is closed under addition (mod 12): for all $x, y \in \mathbb{Z}_{12}$, $x +_{12} y \in \mathbb{Z}_{12}$.
- Addition (mod 12) is associative.
- The identity is 0.
- The inverse of x is $12 - x$.

This group is also abelian since for all $x, y \in \mathbb{Z}_{12}$, $x +_{12} y = x +_{12} x$.

Example 2.6. The **dihedral groups** represent the symmetries of a regular polygon that map it onto itself. Consider the regular hexagon. Let r denote the rotation of through $\pi/3$ about the axis of symmetry perpendicular to the hexagon (rotating), let s denote the rotation through π about an axis of symmetry that lies in the plane of the plate (flipping), and let e denote the identity (leaving the hexagon unchanged). Then the dihedral group D_6 consists of the following 12 twelve elements, as shown in Figure 1.

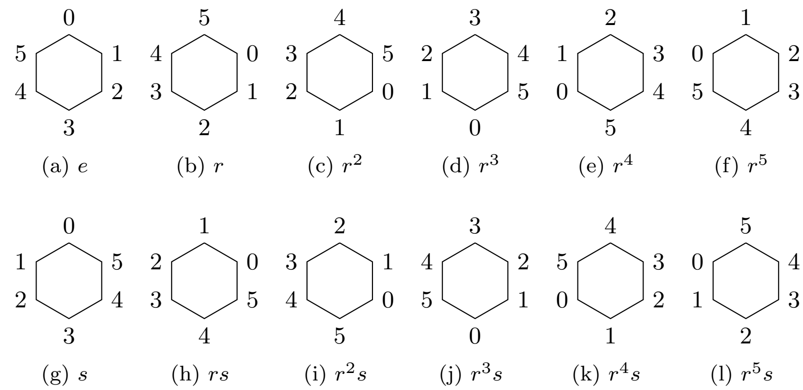


Figure 1: Elements of D_6

It may seem that the group is not closed under multiplication since the element sr is missing from the group. However, a rotation r takes the hexagon from Figure 1a to Figure 1b. A subsequent flip s takes the hexagon from Figure 1b to Figure 1l. Thus, rs is equivalent to r^5s. The reader can check that the whole group is indeed closed under multiplication. In general,

$$sr^n = r^{6-n}s, \qquad \text{for } n \in \mathbb{Z}_6$$

for the D_6 dihedral group.

Definition 2.7. The **order** of a group is the number of elements in the group.

Definition 2.8. The **order** of some element x of a group G is the smallest positive integer n such that $x^n = e$

Definition 2.9. A **subgroup** of a group G is a subset of G which itself forms a group under the multiplication of G

Definition 2.10. A group G is **cyclic** if there exists an $x \in G$ such that for all $y \in G$, $y = x^n$ for some $n \in \mathbb{Z}$. We call x a **generator** of G.

2.2 Permutations

Definition 2.11. A **permutation** of an arbitrary set X is a bijection from X to itself.

Permutations can be denoted in multiple ways. Consider r from the D_6 dihedral group. We can represent it as a permutation of integers like so: (054321), where each integer is sent to the one following it, and the final one is sent to the first. Likewise, we can write sr as (01)(25)(34).

Definition 2.12. A permutation of the form $(a_1a_2 \ldots a_k)$ is called a **cyclic permutation**. A cyclic permutation of length k is called a **k-cycle**.

Definition 2.13. A **transposition** is 2-cycle.

Any k-cycle $(a_1a_2 \ldots a_k)$ can be written as a product of transpositions:

$$(a_1a_2 \ldots a_k) = (a_1a_k) \ldots (a_1a_3)(a_1a_2)$$

Note that transpositions may be written as many different products. This product is not unique, but is meant to show the existence of a product consisting only of transpositions.

Definition 2.14. An **even permutation** is a permutation that can be written as an even number of transpositions. The others are called **odd permutations**.

It may bother the reader that the a permutation in the form of a product of transpositions is not unique. Perhaps a permutation could be written as both an even number of transpositions and an odd number. However, the following theorem shows that the definition is well defined.

Theorem 2.15. *Although any permutation can be written as a product of transpositions in infinitely many different ways, the number of transpositions which occur is always even or always odd.*

Theorem 2.16. *Consider the set X of n elements. The set of all permutations of X forms a group S_n called the* ***symmetric group*** *of degree n. Multiplication on this group is defined by composition of functions.*

2.3 Morphisms

2.3.1 Homomorphisms

Definition 2.17. Let G and G' be groups. A **homomorphism** is a function $\varphi : G \to G'$ that preserves the multiplication of G. Therefore,

$$\varphi(xy) = \varphi(x)\varphi(y) \qquad \text{for all } x, y \in G$$

Example 2.18. Let φ be a function from D_{12} to $\mathbb{Z}_2$ defined by $\varphi(r^n) = 0$ and $\varphi(r^n s) = 1$. Consider two elements x and y in D_{12}. We have four cases:

- $x = r^m$, $y = r^n$:

$$\varphi(xy) = \varphi(r^m r^n) = \varphi(r^{m+n}) = 0 = 0 + 0 = \varphi(x)\varphi(y)$$

- $x = r^m s$, $y = r^n$:

$$\varphi(xy) = \varphi(r^m s r^n) = \varphi(r^{m-n} s) = 1 = 1 + 0 = \varphi(x)\varphi(y)$$

- $x = r^m$, $y = r^n s$:

$$\varphi(xy) = \varphi(r^m r^n s) = \varphi(r^{m+n} s) = 1 = 0 + 1 = \varphi(x)\varphi(y)$$

- $x = r^m s$, $y = r^n s$:

$$\varphi(xy) = \varphi(r^m s r^n s) = \varphi(r^{m-n} ss) = \varphi(r^{m-n}) = 0 = 1 + 1 = \varphi(x)\varphi(y)$$

Hence, φ satisfies the properties of a homomorphism.

2.3.2 Isomorphisms

Definition 2.19. An **isomorphism** is a bijective homomorphism.

Example 2.20. $\mathbb{Z}_{12}$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_4$. Consider the elements of $\mathbb{Z}_3 \times \mathbb{Z}_4$:

(0,0)	(0,1)	(0,2)	(0,3)
(1,0)	(1,1)	(1,2)	(1,3)
(2,0)	(2,1)	(2,2)	(2,3)

Send each element (x, y) to $4x + y$ and you have $\mathbb{Z}_{12}$

2.3.3 Automorphisms

Definition 2.21. An **automorphism** of a group G is an isomorphism from G to G. The set of all automorphisms forms a group under composition of functions, which is called the **automorphism group** of G and written $\text{Aut}(G)$.

Automorphisms fix the identity and send generators to generators.

Example 2.22. Consider the automorphisms of $\mathbb{Z}_4$. There are only two generators in this group: 1 and 3. Therefore, there are only two elements in $\text{Aut}(\mathbb{Z}_4)$: the trivial one, and the one that flips 1 and 3.

2.4 Products

2.4.1 Direct Products

Theorem 2.23. *The set $G \times H$ of two groups G and H is a group that consists of the elements (g, h) where $g \in G$ and $h \in H$. Given two elements (g, h) and (g', h') of $G \times H$, multiplication on this group is defined by*

$$(g, h)(g', h') = (gg', hh')$$

*where the first term, gg', inherits the multiplication of G, and the second, hh', inherits the multiplication of H. We call this group the **direct product** $G \times H$ of G and H.*

Proof. Associativity follows from the associativity in both G and H. The identity is (e, e) and the (g^{-1}, h^{-1}) is the inverse of (g, h). □

Example 2.24. Consider $\mathbb{Z}_4 \times \mathbb{Z}_2$. This group has 8 elements:

(0,0),	(1,0),	(2,0),	(3,0),
(0,1),	(1,1),	(2,1),	(3,1).

Multiplication is defined by

$$(x, y) + (x', y') = (x +_4 x', y +_2 y')$$

2.4.2 Semidirect Products

Theorem 2.25. *Suppose we have the groups G, H and the homomorphism $\varphi : G \to Aut(H)$. Then the "twisted" direct product $G \rtimes H$ forms a new group. Its elements are of the form (g, h) with $g \in G$ and $h \in H$ and multiplication is defined by*

$$(g, h)(g', h') = (g \cdot \varphi(h)(g'), h \cdot h').$$

We call this group the ***semidirect product*** *of G and H.*

Example 2.26. Consider the semidirect product $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$. The elements in this group are the same as those in $\mathbb{Z}_4 \times \mathbb{Z}_2$ as listed in Example 2.24.

We need to define φ. There are only two automorphisms of $\mathbb{Z}_4$ as shown in Example 2.22. Let the trivial automorphism be denoted with e and the other with σ. Then, since $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(H)$, we have $\varphi(0) = e$ and $\varphi(1) = \sigma$.

Now we can perform multiplication on the group. Consider multiplying the elements $(0, 0)$ and $(1, 0)$. Since $\varphi(0) = e$ this multiplication is just like that of the direct product:

$$(0, 0)(1, 0) = (0 + \varphi(0)1, 0 + 0) = (0 + e(1), 0 + 0)(1, 0)$$

Now consider the elements $(1, 0)$ and $(0, 1)$.

$$(0, 1)(1, 0) = (0 + \varphi(1)(1), 1 + 0) = (0 + \sigma(1), 1 + 0) = (3, 1)$$

Example 2.27. We can rewrite the elements of $\mathbb{Z}_4$ as e, r, r^2 and r^3 and the elements of $\mathbb{Z}_2$ as e and s. Converting the elements we used in the above example, we have $(0, 0) = e$, $(1, 0) = r$ and $(0, 1) = s$. Then,

$$(0, 0)(1, 0) = er = r$$

and

$$(0, 1)(1, 0) = sr = \sigma(s)(r)s = r^3s$$

Note how the multiplication is "twisted". That is, to "pull" the r past the s, we have to twist and change it a little bit. We can clearly see that $\mathbb{Z}_4 \times \mathbb{Z}_2$ is isomorphic to the dihedral group D_4.

2.4.3 Wreath Products

The generalized form of a wreath product $G \wr H$ is too complicated for the scope of this paper. Therefore, we will only consider the special case taking a wreath product with $\mathbb{Z}_2$.

Consider the group G. Then $G \wr \mathbb{Z}_2$ is isomorphic to $(G \times G) \rtimes \mathbb{Z}_2$. We will discuss this semidirect product in the dihedral notation as in Example 2.27. Thus, let the elements of $G \times G$ be denoted by $r_1^m r_2^n$.

The automorphism for a wreath product must permute the parts of an element in $G \times G$. Since we only have two elements, r_1^m and r_2^n, the only non-trivial automorphism is to switch them. That is $\sigma(r_1^m r_2^n) = r_2^n r_1^m$. Thus,

$$r_1^m r_2^n s = s\sigma(r_1^m r_2^n) = sr_2^n r_1^m.$$

3 Music Theory

Music theory is a tool and framework with which we explain our listening experience. However, both the tool and the term "listening experience" are loosely defined. They are dependent on the music.

During the mid-15th century, composers began constructing their pieces around a particular pitch, called the tonic. This pitch was quickly established at the start of the piece and all other pitches were heard relative to it. Intervals and chords were labeled as consonant or dissonant. A feeling of tension occurred in various ways, such as when resolution was delayed, or when the music leapt to distant keys (more than two accidentals removed from the tonic). Resolution to the tonic was crucial to ending the piece. After over two centuries of tonal music, listeners have begun to expect music to resolve in particular ways.

Along with the development of tonal music was the development of tonal theory. Its structure and notation allowed theorists to describe the listener's expectation. Thus, it provided an explanation for our reaction to particular harmonies. It explained our feeling of surprise at a particular chord and our feeling of finality at the end of a piece.

Around the turn of the 19th century, composers pushed the boundaries of tonal music. They began using dissonant chords with unprecedented freedom and resolved them in new ways. Eventually, their pieces no longer fit the framework of tonal music. Tonal theory no longer provided an adequate explanation for our listening experience. Thus, a new framework was constructed called atonal music theory.

Discussions in music require a certain vocabulary. The following terms are defined in the appendix:

- interval
- half step (semitone) & whole step (whole tone)
- flat, sharp, natural & accidental
- enharmonic equivalence
- major, minor, mode
- parallel & relative
- scale degree
- triad

3.1 Basic concepts of atonal music theory

Atonal music is based on sequences of pitches and intervals. No particular pitch is considered more important than the others and resolution of dissonance is unimportant. It assumes octave and enharmonic equivalence.

Definition 3.1. Pitches that are separated by an integer multiple of an octave, or are enharmonically equivalent belong to the same **pitch class**.

Each pitch class is assigned an integer as shown in Table 1.

Table 1: Pitch-Class Integers

Integer	Possible Notation
0	B♯, C, D♭♭
1	C♯, D♭
2	C×, D, E♭♭
3	D♯, E♭
4	D×, E, F♭
5	E♯, F, G♭♭
6	F♯, G♭
7	F×, G, A♭♭
8	G♯, A♭
9	G×, A, B♭♭
10	A♯, B♭
11	A×, B, C♭

Pitch-classes are separated by intervals. In atonal music theory, they consider four different definitions of "interval" to aid analysis. However, we will give only one:

Definition 3.2. Consider the pitches a and b. The **ordered pitch-class interval** from a to b is $a - b \pmod{12}$.

Definition 3.3. A **pitch-class set** is an unordered set of pitch-classes, denoted as a string of integers enclosed in brackets. Within a pitch-class set, we do not have information about the register, rhythm or order of the pitches.

Example 3.4. The C major triad consists of the notes C, E and G. This can be represented as the pitch-class set [047], since C = 0, E = 4 and G = 7 as shown in Table 1.

In atonal music, operations are performed on pitch-class sets, creating new pitch-class sets that are spread throughout the music. Thus, the music sounds random and yet structured at the same time. We will discuss two types of operations in this paper: the transpositions and the inversions.

3.1.1 T_n, the Transpositions

Definition 3.5. The transposition T_n moves a pitch-class or pitch-class set up by $n \pmod{12}$. (*Note:* moving down by n is equivalent to moving up by $12-n$.)

Figure 2 from page 34 of [4] shows two lines from Schoenberg's String Quartet No. 4, which are related by T_6. Note how the pitch-class intervals between notes remains do not change. Figure 3 from page 35 of [4] shows a more complicated analysis of Webern's Concerto for Nine Instruments, Op. 24, second movement, which depicts four different pitch-class sets related by different transpositions.

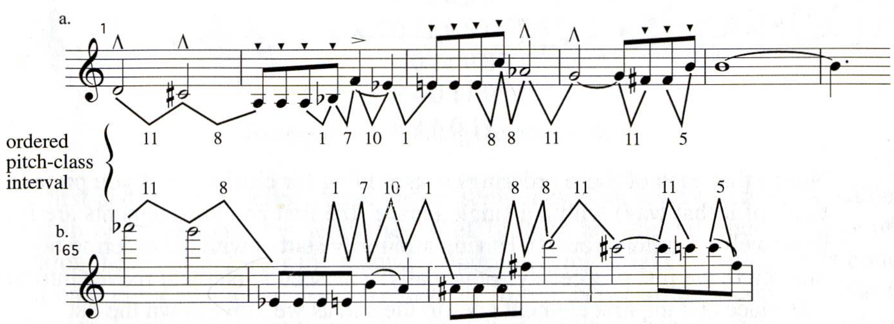


Figure 2: Two lines of pitch classes related by T_6 (Schoenberg, String Quartet No. 4).

3.1.2 T_nI, the Inversions

Definition 3.6. Consider the pitch a. Inversion T_nI inverts the pitch about C (0) and then transposes it by n. That is, $T_nI(a) = -a + n \pmod{12}$.

Figure 4 from page 40 of [4] shows two lines from Schoenberg's String Quartet No. 4, which are related by T_9I. Note how the pitch-class intervals of size n have all been inverted to $12 - n$. Figure 5 from page 41 of [4] shows an analysis of the Schoenberg, Piano Piece, Op. 11, No. 1, which depicts three different pitch-class sets related by different inversions.

4 Group Theory as a Structure for Atonal Music Theory

The numbering of the pitch classes reveals their isomorphism to $\mathbb{Z}_{12}$. More interestingly, the group of transpositions and inversions, denoted T_n/T_nI is isomorphic to the dihedral group D_{12}.

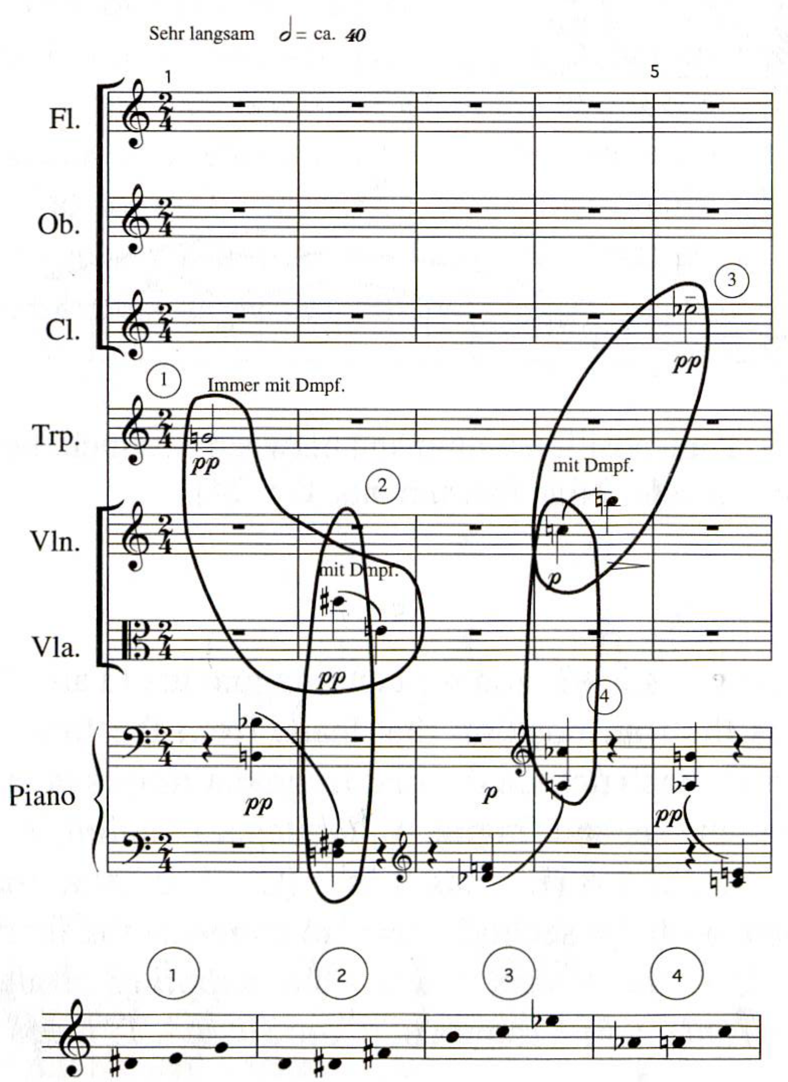


Figure 3: Transpositionally equivalent pitch-class sets (Webern, Concerto for Nine Instruments, Op. 24).

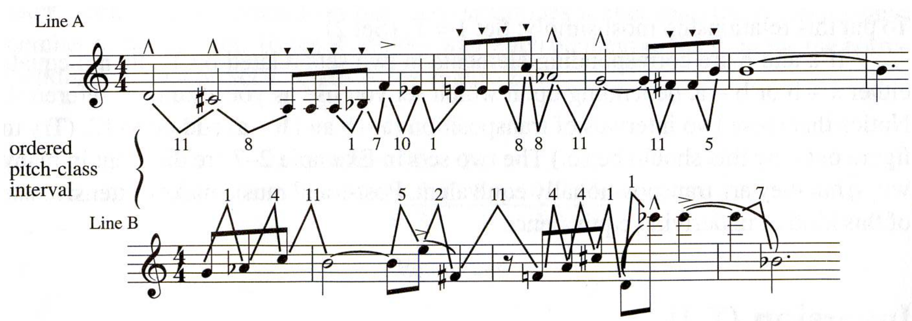


Figure 4: Two lines of pitch classes related by T_9I (Schoenberg, String Quartet No. 4)

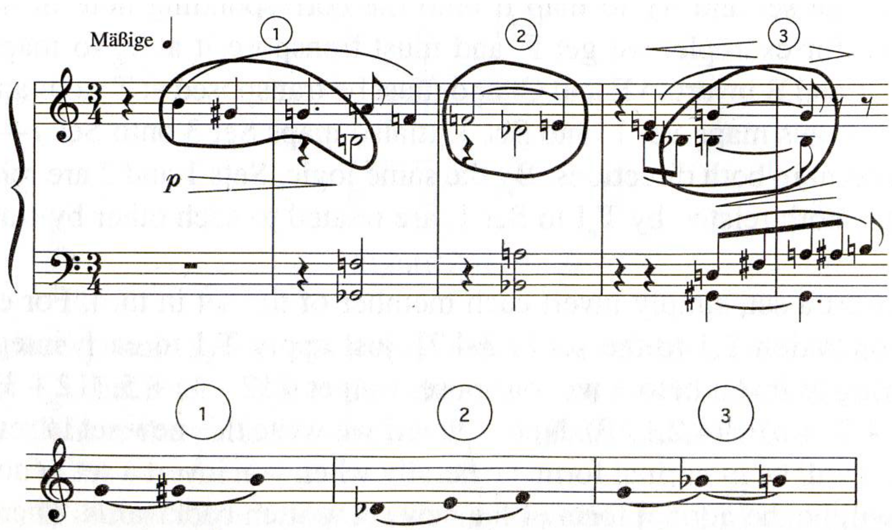
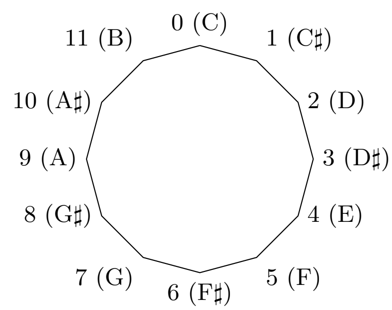
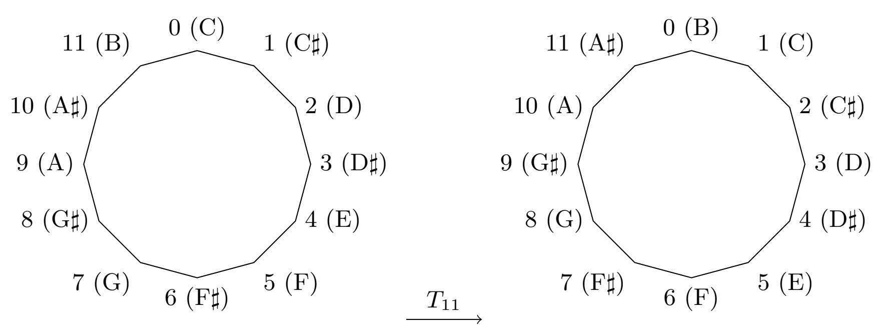


Figure 5: Inversionally equivalent pitch-class sets (Schoenberg, Piano Piece, Op. 11, No. 1)

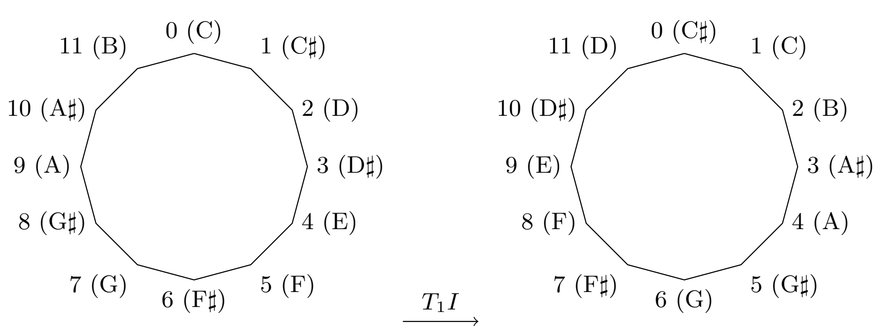
Imagine laying out all the pitches in a circular pattern on a 12-sided polygon. Then we have:



Consider the transposition T_{11}. It sends C to B, C♯to C, etc. That is,



Now consider the inversion T_1I. It sends C to D♯, D♯to C, D to B, etc. This gives:



Note the striking similarity of T_{11} and T_1I to r and sr of D_{12}. We can clearly see that the T_n/T_nI group is isomorphic to D_{12} by $T_n \to r^{12-n}$ and $T_nI \to r^n s$.

5 A Flaw in Atonal Music Theory

Consider the operation that changes a major triad to a minor triad. To our ears, this is the same operation regardless of whether we start with a CM triad or a DM triad. However, consider the CM triad as the pitch class [047], cm as [037], DM as [269] and dm as [259]. Then, we have

$$\text{CM} \to \text{cm} = [047] \to [037] = T_7 I$$
$$\text{DM} \to \text{dm} = [269] \to [259] = T_{11} I$$

This is a misleading representation of the music, because our ears do not hear two different actions. Therefore, the structure of atonal music theory has an inherent flaw. It cannot support these simple transformations.

A music theorist named Hugo Riemann recognized this problem. He invented the idea of a "triadic transformation." Later music theorists devised three operations, called Neo-Riemannian operations, that functioned specifically on triads:

- The Parallel operation (P) moves the middle note of a triad up or down a semitone such that a major triad becomes minor and a minor triad becomes major. For example, it would move the E in a CM to an E♭ and the E♭ in a cm triad to an E♮.

- The Leading-tone exchange (L) moves the bottom note of a major triad down a semitone and the top note of a minor triad up a semitone. Thus, a CM triad would turn into an em triad, and a cm triad would turn into an A♭M triad.

- The Relative operation (R) sends a chord to its relative counterpart by moving the top note of a major triad up by a whole tone, and moving the bottom note of a minor triad down by a whole tone. Thus, a CM triad would turn into an am triad, and a cm triad would turn into an E♭M triad.

These three were particularly interesting because they allowed for parsimonious voice-leading. That is, in moving from one triad to another, only one voice (top, middle or bottom) moved, and it moved by nothing more than a whole step. In addition, they allowed a transformation from any one chord to another by composition of these operations.

6 Uniform Triadic Transformations

This P, L and R notation, while a definite imporvement, could still be unclear, unwieldy and limited in its usefulness. For example, a move from a CM triad to a b♭m triad requires a minimum of six Neo-Rimannian operations. Furthermore, there are nine different ways to write it in six operations: LPRPR, LRPRP, PLRLR, PRLRP, PRPRL, RLPLR, RLRLP, RPLPR, RPRPL. Of course, there are even more ways to write it in more than six operations. Not only has this notation become pedantic, it also fails to reflect the music: who would hear six operations in a simple move from CM to b♭m?

To resolve this problem, another music theorist named Julian Hook devised a new notation for transformations on triads, which he called uniform triadic transformations (UTTs). This notation, in fact, was a group structure with intriguing algebraic properties. Before we jump into a discussion on Hook's UTTs, let us first provide some definitions.

Definition 6.1. A **triad** is an ordered pair $\Delta = (r, \sigma)$ where r is the root of the triad expressed as an integer (mod 12), and σ is a sign representing its mode ($+$ for major, $-$ for minor).

Example 6.2. $\Delta = (0, +)$ represents a C major triad and $\Delta = (6, -)$ represents an f♯ minor triad.

Theorem 6.3. *The set of all 24 major and minor triads, forms a abelian group with multiplication defined by*

$$(t_1, \sigma_1)(t_2, \sigma_2) = (t_1 + t_2, \sigma_1\sigma_2)$$

We call this set Γ.

Proof. This group is clearly isomorphic to $\mathbb{Z}_{12} \times \mathbb{Z}_2$, which by Theorem 2.23 is a group. □

Definition 6.4. Given $\Delta_1 = (r_1, \sigma_1)$ and $\Delta_2 = (r_2, \sigma_2)$, the **transposition level** $t = r_2 - r_1$ is the interval between the roots and the **sign factor** $\sigma = \sigma_1\sigma_2$ is the change in sign. (σ is multiplied as expected with $++ = +, +- = -, -+ = -$ and $-- = +$.) The **Γ-interval** $\text{int}(\Delta_1, \Delta_2)$ is the ordered pair (t, σ) where t and σ are the transposition level and sign factor as defined above.

Example 6.5. The Γ-interval from $(0, +)$ (C major) to $(6, -)$ (f♯ minor) is $(6, -)$.

6.1 Introduction to Triadic Transformations

Definition 6.6. A **triadic transformation** is a bijective mapping from Γ to itself. In other words, it is a permutation of Γ.

Theorem 6.7. *The set of all triadic transformations forms a group* $\mathcal{G}$ *.*

Proof. After numbering the triads, this group is clearly isomorphic to S_{24}. □

The order of $\mathcal{G}$ is huge: 24 factorial. However, most of these transformations have little musical meaning since the action of a transformation on one triad may not resemble its action on another triad.

6.2 $\mathcal{V}$, the Uniform Triadic Transformations

Of particular musical interest are the UTTs because they operate on all major triads in the same manner. Similarly, the UTTs have one action for all minor triads.

Definition 6.8. Consider the triadic transformation that transforms (r, σ) to (r', σ'). It is a **uniform triadic transformation** (UTT) if it transforms $(r+t, \sigma)$ to $(r'+t, \sigma')$ for all $t \in \mathbb{Z}_{12}$.

It is important to note that not *all* musically interesting transformations are UTTs. The inversions T_nI, for example, are not part of the UTTs.

Any UTT is completly determined by three parameters:

- t^+, its transposition level for a major triad
- t^-, its transposition level for a minor triad
- σ, its sign. (*Note:* it may seem that σ could be different for major and minor triads. However, in order to be a transformation, a UTT must map Γ to itself. Thus, if it switches major triads to minor, it must switch minor triads to major. That is $\sigma^+ = -\sigma^-$. Positive σ implies no change in mode (it is **mode-preserving**), negative σ implies switching to the opposite mode (it is **mode-reversing**).)

We can thus denote any UTT U by the ordered triple $U =< \sigma, t^+, t^- >$.

Example 6.9. We will convert Riemann's P, L and R in UTT notation:

$$\begin{aligned} P &=< -, 0, 0 > \\ L &=< -, 4, 8 > \\ R &=< -, 9, 3 > \end{aligned}$$

Note, Hook uses left-to-right orthography. Thus, $U1U2$ implies "first $U1$ then $U2$." As usual, $U^2 = UU$, etc. Although it is less intuitive for mathematicians, we will adhere to his notation.

6.2.1 Multiplication on $\mathcal{V}$

Multiplication on $\mathcal{V}$ should clearly be composition. Before we derive a general formula for the composition of two UTTs, let us consider some concrete examples.

Example 6.10. Consider the UTTs $U =< +, 4, 7 >$ and $V =< -, 5, 10 >$. Let us calculate the product $UV =< \sigma_{UV}, t^+_{UV}, t^-_{UV} >$. When UV acts on a CM triad ($\Delta = (0, +)$) we have:

$$(0,+) \xrightarrow{U} (4,+) \xrightarrow{V} (9,-).$$

Thus, UV transforms the major triads through the Γ-interval $(9, -)$. We can deduce that $\sigma_{UV} = -$ and $t^+_{UV} = 9$.

When UV acts on a cm triad ($\Delta = (0, -)$) we have:

$$(0,-) \xrightarrow{U} (7,-) \xrightarrow{V} (5,+).$$

Thus, UV transforms the minor triads through the Γ-interval $(5,+)$ and $t^-_{UV} = 5$. Hence, $UV =< -, 9, 5 >$

This product may be calculated by multiplying the signs ($\sigma_{UV} = \sigma_U \sigma_V$) and adding the corresponding transposition levels ($t^+_{UV} = t^+_U + t^+_V$ and $t^-_{UV} = t^-_U + t^-_V$). Figure 6 from page 72 of [3] depicts a visual representation of UV.

Example 6.11. Now consider the product VU. In this case we have

$$(0,+) \xrightarrow{V} (5,-) \xrightarrow{U} (0,-),$$

and

$$(0,-) \xrightarrow{V} (10,+) \xrightarrow{U} (2,+).$$

Therefore, $VU =< -, 0, 2 >$. In this case, the signs were multiplied as before, the transposition levels were "cross-added." That is, $t^+_{UV} = t^+_V + t^-_U$ and $t^-_{UV} = t^-_V + t^+_U$.

We can see that in the above example, the "cross-adding" was due to the sign of the first transformation. In the first case, the first UTT (U) was mode-preserving, so the second UTT (V) acted on the same mode as U. Thus, the corresponding transposition levels were applied in succession. In the second case, the first UTT (V) was mode-reversing, so the second UTT (U) acted on the *opposite* mode as V and *opposite* transposition levels were combined. This leads us to the general form of UTT multiplication:

Theorem 6.12. *Consider two UTTs $U =< \sigma_U, t^+_U, t^-_U >$ and $V =< \sigma_V, t^+_V, t^-_V >$. Multiplication on $\mathcal{V}$ is given by*

$$UV =< \sigma_U \sigma_V, t^+_U + t^{(\sigma_U)}_V, t^-_U + t^{(-\sigma_U)}_V >$$

The reader can verify that following the process above using two arbitrary elements in $\mathcal{V}$ will give the desired result.

6.2.2 Inversion on $\mathcal{V}$

Again, before we derive a general formula for the inverse of a UTT, let us consider some concrete examples.

Example 6.13. Consider the UTT $U =< +, 4, 7 >$. Because $(0,+) \xrightarrow{U} (4,+)$ and $(0,-) \xrightarrow{U} (7,-)$, we need $(4,+) \xrightarrow{U^{-1}} (0,+)$ and $(7,-) \xrightarrow{U^{-1}} (0,-)$. Thus, $U^{-1} =< +, 8, 5 >$. Note how this is simply the inversion of the transposition levels: $< +, -4 \pmod{12}, -7 \pmod{12} >$.

Example 6.14. Now consider the UTT $V =< -, 5, 10 >$. $(0,+) \xrightarrow{V} (5,-)$ and $(0,-) \xrightarrow{V} (10,+)$. Thus, $(5,-) \xrightarrow{V^{-1}} (0,+)$ and $(10,+) \xrightarrow{V^{-1}} (0,-)$. Therefore, $V^{-1} =< -2, 7 >$ or $< -, -10 \pmod{12}, -5 \pmod{12} >$. In this case, the transposition levels are not only inverted, but interchanged. Once again, this is due to the sign of V.

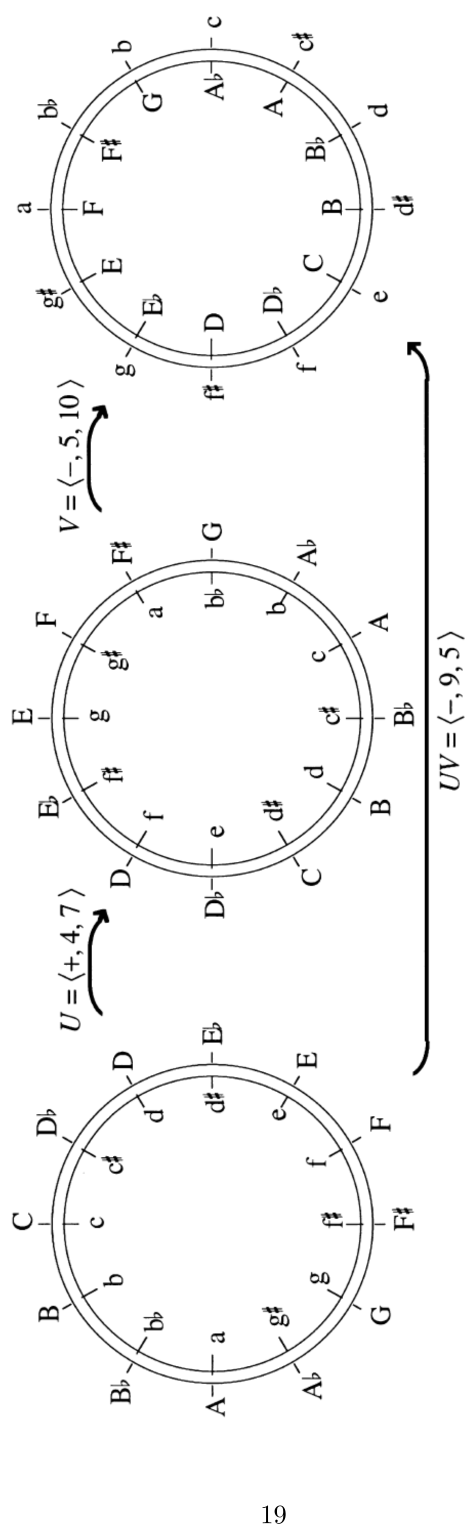


Figure 6: Visual representation of the UTT equation $< +, 4, 7 >< -, 5, 10 >=< -, 9, 5 >$

Similarly to multiplication, the inverse of a UTT can be derived to be as follows:

Theorem 6.15. *Consider the UTT* $U =< \sigma, t^+, t^- >$. *Its inverse is given by*

$$U^{-1} =< \sigma, -t^{\sigma}, -t^{(-\sigma)} >$$

6.2.3 Isomorphism to $\mathbb{Z}_{12} \wr \mathbb{Z}_2$

With some analysis, we come upon the highly interesting and significant result that $\mathcal{V}$ is isomorphic to $\mathbb{Z}_{12} \wr \mathbb{Z}_2$. This isomorphism gives $\mathcal{V}$ any and all results already proven about $\mathbb{Z}_{12} \wr \mathbb{Z}_2$.

Theorem 6.16. *The set* $\mathcal{V}$ *of UTTs is a group that is isomorphic to* $\mathbb{Z}_{12} \wr \mathbb{Z}_2$.

Proof. Remember from Section 2.4.3 how we represented $G \times G$ in the form $r_1^m r_2^n$. Let G be $\mathbb{Z}_{12}$ and let us switch the 1 and 2 to + and -. Then the elements of $\mathbb{Z}_{12} \wr \mathbb{Z}_2$ are $sr_+^m r_-^n$ where $m, n \in \mathbb{Z}_{12}$.

Also note that the transpositions, which are equivalent to the transposition levels of a UTT, are isomorphic to the rotations of the D_{12} by $T_n \to r^n$. (*Note:* we initially wrote in Section 4 that the T_n/T_nI group was isomorphic by $T_n \to r^{12-n}$. This reflected our visual representation of the two groups. However, the groups are still isomorphic if we choose $T_n \to r^n$, which makes the following argument clearer.)

It seems likely that the UTTs are isomorphic to $\mathbb{Z}_{12} \wr \mathbb{Z}_2$ by $< \sigma, t^+, t^- > \to sr_+^m r_-^n$. We have already shown that the UTTs have multiplication and inverses. We only need to show that it follows the multiplication of $\mathbb{Z}_{12} \wr \mathbb{Z}_2$. That is, $(r_+^m r_-^n)s = s(r_-^n r_+^m)$.

$$\begin{aligned}
(r_+^m r_-^n)s &= (er_+^m r_-^n)(see) \\
&=< -, 0, 0 >< +, t_m^+, t_n^- >, \\
&=< -, 0 + t_n^-, 0 + t_m^+ > \\
&=< -, t_n^-, t_m^+ > \\
&= s(r_-^n r_+^m)
\end{aligned}$$

this is a little backwards because mathematicians use right-to-left orthography and Hook uses left-to-right orthography

Thus, multiplication is preserved and the isomorphism holds. □

6.2.4 Even and Odd UTTs

UTTs can be classified as "even" or "odd" in multiple ways. Let us begin with "even/odd in the sense of total transposition."

Definition 6.17. We say that a UTT $U =< \sigma, t^+, t^- >$ is **even** (or, more fully, **even in the sense of total transposition**) if its total transposition $\tau(U) = t^+ + t^-$ is an even number. U is **odd (in the sense of total transposition)** if $\tau(U)$ is an odd number.

Example 6.18. The UTTs $P = < -, 0, 0 >$, $L = < -, 4, 8 >$ and $R = < -, 9, 3 >$ are all even. Also, the Riemannian UTTs, whose total transposition τ is 0 by definition, are all even.

Note that the UTTs can be written as a permutation of the 24 triads. Take for example, the UTT $U = < -, 0, 8 >$, which breaks down into four 6-cycles:

C	→	c	→	A♭	→	g♯	→	E	→	e	→	C,
D♭	→	c♯	→	A	→	a	→	F	→	f	→	D♭,
D	→	d	→	B♭	→	b♭	→	F♯	→	f♯	→	D,
E♭	→	d♯	→	B	→	b	→	G	→	g	→	E♭.

We can represent this in the more compact form of:

(C, c, A♭, g♯, E, e)(D♭, c♯, A, a, F, f)(D, d, B♭, b♭, F♯, f♯)(E♭, d♯, B, b, G, g).

Recall from Section 2.2 that each of these 6-cycles can be represented as a product of transpositions (the 2-cycles). Accordingly, we have a new definition of even or odd:

Definition 6.19. A UTTs is **even (in the sense of permutation theory)** if it can be written as a product of an even number of 2-cycles and **odd (in the sense of permutation theory)** if it can be written as a product of an odd number of 2-cycles.

It is remarkable that the two definitions of even or odd (in the sense of total transposition versus permutation theory) are actually equivalent.

Theorem 6.20. *A UTT is even in the sense of total transposition if and only if it is even in the sense of permutation theory.*

The proof of the theorem is given on page 97 of [3]

6.3 $\mathcal{R}$, the Riemannian UTTs

Recall the Neo-Riemannian operators P, L and R as introduced in Section 5 and written as UTTs in Example 6.9. For each, the transposition level for a major triad is equal and opposite that of a minor triad. We define the Riemannian UTTs as follows.

Definition 6.21. A **Riemannian UTT** is a UTT such that $t^+ = -t^-$.

Theorem 6.22. *The set of $\mathcal{R}$ of Riemannian UTTs is isomorphic to D_{12}.*

Proof. D_{12} can be defined as the group of order 24 generated by s and r, such that $s^2 = e$, $r^{12} = e$ and $sr = r^{-1}s$.

The generators of $\mathcal{R}$ are $< -, 0, 0 >$ and $< +, 1, 11 >$.

$$\begin{aligned}
(< -, 0, 0 >)^2 &=< -, 0+0, 0+0 >=< +, 0, 0 >= e \in \mathcal{R} \\
(< +, 1, 11 >)^{12} &=< +, 12 \cdot (1), 12 \cdot (11) >=< +, 0, 0 >= e \\
(< +, 1, 11 >)^{-1} &=< +, 11, 1 > \\
< -, 0, 0 >< +, 1, 11 > &=< -, 0+11, 0+1 > \\
&=< -, 11+0, 1+0 > \\
&=< +, 1, 11 >< -, 0, 0 > \\
&= r^{-1}s
\end{aligned}$$

□

6.4 $\mathcal{K}$, the Subgroups of $\mathcal{V}$

Generally, it is difficult to list all the subgroups of a given group G. However, it is possible to list all the subgroups of $\mathcal{V}$.

Definition 6.23. Give two integers a and b (mod 12), we define three subsets of $\mathcal{V}$ as follows.

- $\mathcal{K}^+(a)$ is the set of all mode-preserving UTTs of the form $< +, n, an >$ as n ranges through the integers mod 12.
- $\mathcal{K}^-(a, b)$ is the set of all mode-reversing UTTs of the form $< -, n, an+b >$.
- $\mathcal{K}(a, b) = \mathcal{K}^+(a) \cup \mathcal{K}^-(a, b)$

Theorem 6.24. *$\mathcal{K}(a, b)$ is a subgroup of $\mathcal{V}$ if and only if the numbers a and b satisfy $a^2 = 1$ and $ab = b$ (mod 12).*

The proof is given on pages 84-85 of [3].

The condition $a^2 = 1$ is satisfied only for $a = 1, 5, 7$ and 11. If $a = 1$ then the condition $ab = b$ is automatically satisfied. For other values of a, the allowable values of b are different in each case. The following is a complete list of the groups $\mathcal{K}(a, b)$:

$$\begin{aligned}
&\mathcal{K}(1,0), \mathcal{K}(1,1), \mathcal{K}(1,2), \ldots, \mathcal{K}(1,11) \\
&\mathcal{K}(5,0), \mathcal{K}(5,3), \mathcal{K}(5,6), \mathcal{K}(5,9) \\
&\mathcal{K}(7,0), \mathcal{K}(7,2), \mathcal{K}(7,4), \mathcal{K}(7,6), \mathcal{K}(7,8), \mathcal{K}(7,10) \\
&\mathcal{K}(11,0), \mathcal{K}(11,6)
\end{aligned}$$

7 Musical Application

The UTTs of order 24 are of considerable musical interest. When such a transformation is applied repeatedly, the resulting chain of triads will cycle through all 24 major and minor triads before returning to the original one. Take, for example, the UTT $U = < -, 9, 8 >$. Its repeated application produces a chain in the scherzo of Beethoven's Ninth Symphony (mm. 143-171):

$$\mathrm{C} \xrightarrow{U} \mathrm{a} \xrightarrow{U} \mathrm{F} \xrightarrow{U} \mathrm{d} \xrightarrow{U} \mathrm{B}\flat \xrightarrow{U} \cdots \xrightarrow{U} \mathrm{A}$$

This chain is 19 triads long, only five short of a complete cycle.

Such triad chains are rarely prolonged to this extent. There are, however, examples from litterature that circumnavigate the entire cycle of 24 triads. These are found in collections of pieces such as Bach's *Well-Tempered Clavier* and the Chopin Preludes, Op. 28. Table 2 from page 90 of [3] lists some other examples of triad chains.

Table 2: Chord Progressions and Tonal Cycles Generated by Order-24 UTTs

Chord progressions		
Source	Progression	UTT(U)
Bach, Violin Concerto in A minor, I, mm. 88-94	e→E→a→A→d→ ···	$< -, 5, 0 >$
Mozart, Requiem, *Confutatis*, mm10–12	C→c→G→g→D→ ···	$< -, 0, 7 >$
Beethoven, String Quartet, Op. 18, No. 6, IV, mm. 20–28	B→e→F♯→b→C♯→ ···	$< -, 5, 2 >$
Beethoven, Symphony No. 3, I, mm. 178–186	c→A♭→c♯→A→d	$< -, 5, 8 >$
Beethoven, Symphony No. 9, II, mm. 143–171	C→a→F→d→B♭→ ···	$< -, 9, 8 >$
Liszt, "Wilde Jagd," mm. 180–184	E♭→g→D→f♯→D♭→ ···	$< -, 4, 7 >$
Tonal cycles		
Source	Key Sequence	UTT(U)
Bach, *Well-Tempered Clavier*	C→c→C♯→c♯→D→ ···	$< -, 0, 1 >$
Chopin, Preludes, Op. 28	C→a→G→e→D→ ···	$< -, 9, 10 >$
Liszt, *Transcendental Etudes*	C→a→F→d→B♭→ ···	$< -, 9, 8 >$

8 Conclusion

We can see that Hook's UTTs not only have interesting mathematical properties, but also musical significance. Musically, the choice of triads is non-arbitrary since triads occur throughout music. Mathematically, however, the set choice is arbitrary. Some current research in music theory investigates the generalization Hook's UTTs to larger chords or even to pitch-class sets. The flexibility these generalizations will provide may greatly improve atonal analysis.

Appendix

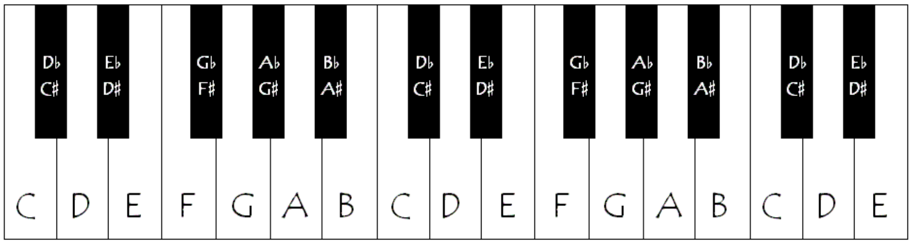


Figure 7: A piano with the keys labeled

Definition 8.1. An **interval** is the distance between two pitches

Definition 8.2. Two pitches have an interval of a **half step** or **semitone** apart if they "touch" each other on a piano. A **whole step** or **whole tone** is two semitones.

Example 8.3. C is a semitone away from C♯and B. It is a whole tone away from D and B♭.

Definition 8.4. A **flat** ♭ lowers a pitch by a semitone, a **sharp** ♯ raises a pitch by a semitone and a **natural** ♮ does nothing to the pitch. A **double flat** ♭♭ lowers a pitch by a whole tone, and a **double sharp** $\times$ raises a pitch by a whole tone. The symbols ♭, ♯ and ♮ are called **accidentals**.

Example 8.5. Flats and sharps of notes are shown in Figure 7. D♭♭is C and D$\times$ is E.

Definition 8.6. If two names refer to the same note on the piano, they are considered **enharmonically equivalent**.

Example 8.7. C♯ is enharmonically equivalent to D♭.

Definition 8.8. A **major scale** has the following sequence of whole steps (W) and half steps (H):

WWHWWWH.

A **minor scale** has the following sequence of whole steps and half steps:

WHWWHWW.

Major is denoted by uppercase letters and minor by lowercase letters. The terms "major" and "minor" denote the **mode** of a scale or other musical object.

Example 8.9. The C major scale has the notes CDEFGABC and the c minor scale has the notes CDE♭FGA♭B♭C.

Definition 8.10. **Scale degree** $\hat{n}$ refers to the n^{th} note of a scale.

Example 8.11. $\hat{3}$ of the C major scale refers to E, and $\hat{3}$ of the c minor scale refers to E♭.

With the definition of scale degrees in hand, we can redefine a minor scale as a major scale with $\hat{3}$, $\hat{6}$, and $\hat{7}$ lowered by a semitone.

Definition 8.12. Each scale uses only one type of accidental (♯ or ♭). They are not mixed within scales. A scale is the **parallel major** scale of a minor scale if it contains the same number and type of accidental and it is the **relative major** scale if it begins on the same note. Similarly for a parallel and relative minor scale.

Example 8.13. The parallel minor of C major is c minor. The relative minor of C major is a minor.

Definition 8.14. A **triad** XM consists of scale degrees $\hat{1}$, $\hat{3}$ and $\hat{5}$ from the X major scale. xm comes from the minor scale.

In general, a **major triad** is a chord with three pitches such that the interval between the lowest and the middle pitch is two whole steps and the interval between the middle and highest pitch is a whole step plus a half step. A **minor triad** is a chord with three pitches such that the intervals between the pitches are opposite those of a major triad.

Example 8.15. CM contains the pitches C, E and G. Cm contains the pitches C, E♭ and G.

Figure 8 shows the piano keys placed onto the music staff for those who are unable to read music.

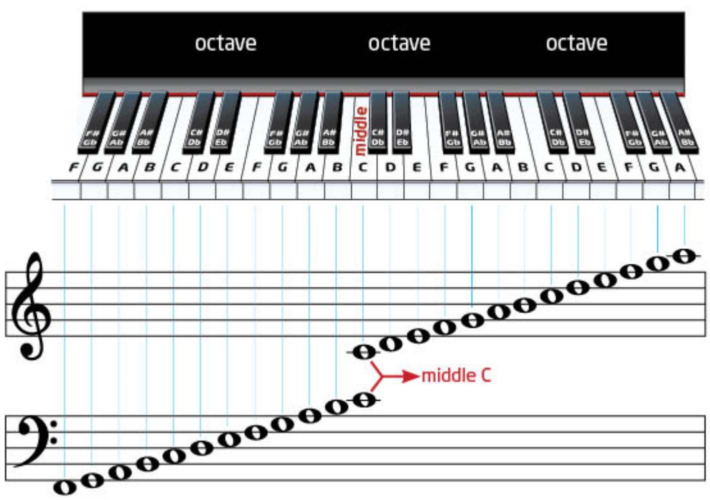


Figure 8: A piano with the keys placed onto the music staff

References

[1] Armstrong, M. *Groups and Symmetry.* Springer-Verlag, 1988.

[2] Dummit, D. S., and Foote, R. M. *Abstract Algebra*, third ed. Wiley, 2003.

[3] Hook, J. Uniform triadic transformations. *Journal of Music Theory 46*, 1-2 (2002), 57–126.

[4] Straus, J. N. *Introduction to Post-Tonal Theory*, second ed. Prentice-Hall, Inc., 2000.