Prime Number Theorem

Note Title 6/2/2010

This is an outline of the Newman-Zagier-Gamelin proof of the Prime Number Theorem (PNT)

<u>Notation</u>

$\pi(x) = \#\text{primes} \le x$, P = set of primes

$$\vartheta(x) = \sum_{\substack{p \le x \\ p \in P}} \log p$$

$$\Phi(s) = \sum_{p \in P} \frac{\log p}{p^s}, \quad Re(s) > 1$$

<u>PNT</u>: $\displaystyle\lim_{x \to +\infty} \frac{\pi(x)\log x}{x} = 1$

<u>Theorem 1.</u> $\displaystyle\lim_{x \to +\infty} \frac{\vartheta(x)}{x} = 1 \iff \lim_{x \to +\infty} \frac{\pi(x)\log x}{x} = 1$

<u>Theorem 2.</u> If $\displaystyle\int_0^\infty \left(\frac{\vartheta(x)}{x} - 1\right)\frac{dx}{x}$ converges

then $\displaystyle\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1.$

<u>Remark</u>: If $\displaystyle\int_0^\infty [\vartheta(e^t)e^{-t} - 1]\,dt$ converges

then $\displaystyle\int_0^\infty \left(\frac{\vartheta(x)}{x} - 1\right)\frac{dx}{x}$ converges.

Let $h(t) = \vartheta(e^t)e^{-t} - 1.$

Then $\mathcal{L}h(s) = \int_0^\infty h(t)e^{-st}\,dt$, for $Re(s)>0$

(1) $$= \frac{\Phi(s+1)}{(s+1)} - \frac{1}{s}.$$

(2) $$-\frac{\zeta'(s)}{\zeta(s)} = \Phi(s) + \sum_{p\in P} \frac{\log(p)}{p^s(p^s-1)}$$

Theorem 3 From (2) $\Phi(s)$ extends meromorphically across $Re(s)=1$ and hence $\mathcal{L}h(s)$ extends meromorphically across $Re(s)=0$

The only possible poles of $\mathcal{L}h(s)$ on $Re(s)=0$ are at zeros of $\zeta(s+1)$ on $Re(s)=0$.

Theorem 4 $\zeta(s)$ has no zeros on $Re(s)=1$.

Corollary $\mathcal{L}h(s)$ extends analytically across $Re(s)=0$.

Theorem 5 (Tauberian theorem) $\mathcal{L}h(s)$ extends analytically across $Re(s)=0$ implies that

$$\int_0^\infty h(t)\,dt = \lim_{\epsilon\to 0^+} \mathcal{L}h(\epsilon)$$

and hence converges.

By the Remark, Theorem 2 and Theorem 1, PNT is true.