Trigonometry on the Complex Unit Sphere

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Contents

1 Introduction

When studying real valued trigonometric functions, the unit circle (henceforth unit sphere) provides a proper and intuitive motivation for the sine, cosine, and tangent functions. A similar process can be used to derive the formulae for the complex trigonometric functions, although the extension of a unit sphere into the two dimensional complex plane is problematic as a concept, in large part due to the 4-dimensional nature of the complex plane.

This paper defines the unit sphere and makes some basic descriptions of its geometry. Additionally, definitions of complex angles and a complex distance operator over the complex plane are defined. With all these details set in place, the paper ends with the derivation of complex trigonometric functions. This method of derivation is often ignored due to its complexity and instead presented with analytic reasoning using Euler's formula. It is, however, helpful in understanding the behavior of the complex trigonometric functions and provides a perhaps more intuitive insight into the behavior of the functions.

1.1 Notation

Let $\mathbb{C}^2$ represent the complex plane represented by the set of vectors $\{(z, w) : z, w \in \mathbb{C}\}$ where $z = x+iy$ and $w = u+iv$. Additionally, borrowing the notation of Hammack [2, p. 211], let U be the complex unit sphere.

1.2 General Unit Spheres

A unit sphere is defined to be the set of points of distance 1 from a fixed point in a general vector space. "The" unit sphere of a vector space typically refers to the unit sphere about the point defined as the origin. Distances can be non-Euclidean, and are perhaps better described as norms. In Euclidean space of n dimensions, the unit sphere is given in equation form by

$$x_1^2 + x_2^2 + \ldots + x_n^2 = 1.$$

This definition is used later to develop the complex unit sphere.

1.3 Geometric Interpretation of Hyperbolic Trigonometry

The real hyperbolic trigonometric functions have a geometric interpretation as parameterizing the so-called unit hyperbola in a similar fashion to how the regular trig functions parameterize the unit circle. In fact, the regular trig functions were traditionally referred to as the circular trig functions to differentiate them from the hyperbolic versions.

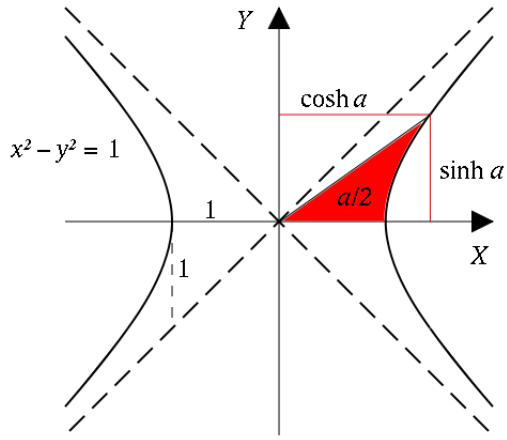

Figure 1: A graph of the unit hyperbola in $\mathbb{R}^2$. The red region serves as a geometric basis for defining sinh and cosh. Image from [3].

Examine the hyperbola in the real plane defined $x^2 - y^2 = 1$. The line from the origin to any point p on the right side of the hyperbola forms the region A additionally bounded by the x axis and the hyperbola, as shown in red in Figure 1. Letting the area of A be denoted by $t/2$, then the coordinates of p are $(\cosh t, \sinh t)$. If p is below the x axis, then $t/2$ is defined as the negative area of A. The left half of the parabola can similarly be parameterized as $(-\cosh t, \sinh t)$.

This is the proper geometric motivation for sinh and cosh. These functions play a critical role in defining and understand the complex unit sphere.

2 Complex Unit Sphere

The complex unit sphere will be defined in $\mathbb{C}^2$ by extending the definition and characteristics of the real unit sphere in $\mathbb{R}^2$.

2.1 Definition

Hammack begins by defining the set U as the set in $\mathbb{C}^2$ such that

$$z^2 + w^2 = 1$$

[2, p. 211]. This equation is identical to that of the real unit circle if z and w are replaced with real numbers; indeed, this shows that U intersects the x, u plane exactly on the real unit circle.

2.2 Complex Distance

This definition of U corresponds to the general definition of a unit sphere in $\mathbb{C}^2$ with a (generalized) difference d defined as

$$d\,(z, w) = \sqrt{(u - x)^2 + (v - y)^2} \tag{1}$$

where the principal branch of the square root is used (defined definitively for z with arguments strictly between $-\pi$ and π and mapping to the right half of $\mathbb{C}$; see [1]). Notice that this is not a formal metric, since complex numbers cannot be compared with inequality operators. This complex distance is, however, the natural extension of Euclidean distance in 2-space.

This construction of U get arbitrarily far from the origin, which is seemingly contradictory to the definition of a unit sphere. Examine, for example, the parts of the unit sphere where z is entirely real and w is entirely imaginary. In equation form, this is

$$x^2 - v^2 = 1,$$

the unit hyperbola in the x, vi plane. The solutions to this can be arbitrarily large, so there exist points in U that have arbitrarily large components. But the distance operator defined in (1) yields that every point on the unit sphere has a distance of exactly 1 from the origin. Indeed, for any $(z, w) \in U$:

$$d\left(\begin{bmatrix} z \\ w \end{bmatrix}, \mathbf{0}\right) = \sqrt{z^2 + w^2} = 1.$$

Therefore U is the unit sphere under the distance operator (1) in the complex plane. Another quirk to the idea of complex distance is that there exists points $\zeta \neq \omega$ such that

$$d(\zeta, \omega) = 0.$$

Take, for example, the points $\zeta = \alpha + i\beta$ and $\omega = \beta - i\alpha$. The distance between these two points defined by this operator is 0, even though they are not equal. Although this is not relevant to the trigonometric functions, it is a curious property of the defined distance operator, and could be used to extend the notion of the origin of $\mathbb{C}^2$ into a set with infinitely many unique, nonzero points.

2.3 Geometry

It has already been stated that U intersects $\mathbb{R}^2$ precisely on the real unit circle. Other "cross-sections" and three dimensional constructs offer additional insights into the geometry of U. Hammack explains that setting only y to be zero yields

$$x^2 + u^2 - v^2 + 2iuv = 1$$

in the x, u, vi space [2, p. 211]. Since the imaginary part of the left side must be zero, either u or v must be zero. Letting $v = 0$ again shows that U intersects $\mathbb{R}^2$

on the real unit circle. Letting $u = 0$ instead shows that U intersects the x, vi plane on the hyperbola $x^2 - v^2 = 1$. An equivalent result with is found when v is the first zeroed variable. Figure two shows this cross-section as a graph.

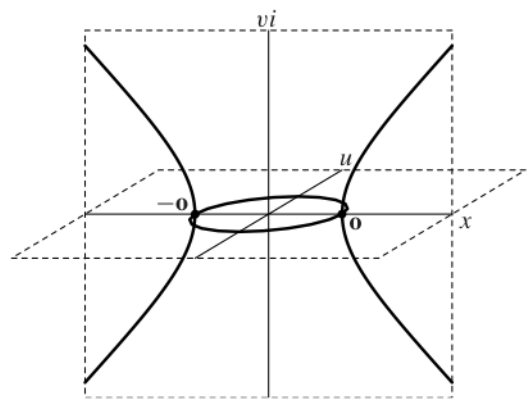

Figure 2: The cross-section of U when v is set to 0. Borrowed directly from [2, p. 211]

Expanding on the method of Hammack, The real components can similarly be set to zero to get another cross-section. Letting $x = 0$ yields

$$-y^2 + u^2 - v^2 + 2iuv = 1.$$

Again, u or v must be 0. However, since u, v, and y are real, there are no solutions if $u = 0$. This implies that $v = 0$ whenever $x = 0$, again showing that U intersects the u, vi plane in a hyperbola and additionally that U never intersects the purely imaginary plane. Indeed, the sum of two squared imaginary numbers is non-positive and therefore never equal to 1.

3 Trigonometry

3.1 Imaginary Trigonometry

Complex angles can be made sense of in an analogous fashion to regular radian angles. Begin by examining the x, u plane and the x, vi plane. U in the x, u plane is simply the real unit circle. Therefore the angle between the lines from the origin to $(1, 0)$ and a point p in U is defined in the traditional fashion as the arc length from $(1, 0)$ to the point p. If this distance is t, then p is parameterized with $(\cos t, \sin t)$.

Extending this definition, now examine U in the x, vi plane. This is a unit hyperbola. Hyperbolas can be parameterized by the hyperbolic sine and cosine,

so if ζ is a point in the right half of the hyperbola, it can be parameterized by $(\cosh t, i \sinh t)$ where t is twice the real area formed by the region defined in Figure 1.

So let the radian angle associated with p be the distance along the hyperbola from $(1, 0)$ to ζ. In Euclidean space, if a curve is given by $f(x)$ where $x \in [a, b]$, then the arc length s from $f(a)$ to $f(b)$ would be measured by integrating

$$s = \int_a^b \sqrt{[f'(x)]^2 + 1}\ dx. \tag{2}$$

However, this is derived using the Pythagorean theorem, which is based on the Euclidean definition of distance. Matching the derivation of (2) (see [4]), let the infinitesimal complex arc length ds be given by

$$\begin{aligned} ds &= \sqrt{dx^2 + (idv)^2} \\ &= \sqrt{1 - \left(\frac{dv}{dx}\right)^2}\ dx. \end{aligned}$$

So s is given by

$$s = i \int_a^b \sqrt{f'(x)^2 - 1}\ dx.$$

Applying this to the upper right portion of the unit hyperbola where $f(x) = \sqrt{x^2 + 1}$, the radian angle θ_ζ is therefore given by

$$\begin{aligned} \theta_\zeta &= i \int_1^{\cosh t} \sqrt{\left(\frac{x}{\sqrt{x^2 - 1}}\right)^2 - 1}\ dx. \\ &= i \int_1^{\cosh t} \frac{1}{\sqrt{x^2 - 1}}\ dx. \end{aligned}$$

This integrand is the known derivative of $\cosh^{-1}$, so the integral evaluates to it (this is similarly derived in [2, p. 214-5]). Notice that this is an imaginary value; the distance along the hyperbola is therefore imaginary, and radian angles along the hyperbola are likewise imaginary. Following the process for the parameterization of the real unit sphere, let the hyperbola-paramaterizing imaginary trig functions be given by

$$\cos it = x = \cosh t, \qquad \sin it = iv = i \sinh t. \tag{3}$$

3.2 Complex Trigonometry

Now, with these bases, it is almost trivial to derive the trigonometric functions for a complex number. It is known that

$$\begin{aligned} \sin(a + b) &= \sin a \cos b + \cos a \sin b \\ \cos(a + b) &= \cos a \cos b - \sin a \sin b. \end{aligned}$$

Since the goal of this paper is to derive the complex trigonometric functions without Euler's formula, it should be noted that the addition forumlae can be derived using purely geometric means, with no complex analysis (see [5]). While Euler's formula yields the most common and simple proof, it is also the most common and simple means of deriving the definition for the complex trigonometric functions. So assuming the geometric proofs is appropriate.

Working under the assumption that the trigonometric functions of complex numbers should behave the same way, this yields that

$$\sin(\alpha + i\beta) = \sin\alpha\cos i\beta + \cos\alpha\sin i\beta$$
$$\cos(\alpha + i\beta) = \cos\alpha\cos i\beta - \sin\alpha\sin i\beta.$$

Using (3), these become

$$\sin(\alpha + i\beta) = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta \tag{4}$$
$$\cos(\alpha + i\beta) = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta, \tag{5}$$

the traditional equations that can be derived from Euler's formula.

If α or β are not positive, then Notice that these equations parameterize U. It can be seen pretty easily that $\cos^2 z + \sin^2 z$ for complex z is equal to 1, just as in the real case. So letting the point (z, w) be parameterized as $(\cos(\alpha + i\beta),\ \sin(\alpha + i\beta))$ trivially yields U.

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