Math 336 Sample Problems

One notebook sized page of notes will be allowed on the test. The test will cover through chapter IV, section 5 in the text.

1. Suppose that v is the harmonic conjugate of u and u is the harmonic conjugate of v. Show that u and v must be constant.

2. Gamelin, IV.5, problem 3.

3. Gamelin, IV.5, problem 4.

4. Let u be harmonic on W. Prove that $f(z) = u_x(z) - iu_y(z)$ is harmonic.

5. Compute $\displaystyle\int_{|z|=3} \frac{dz}{(z-1)(z-2)(z-4)}$.

6. Give an example of a function f for which $\int_{|z|=r} f(z)dz = 0$ for all $r > 0$ although f is not analytic.

7. Let $f(z) = u(x,y) + iv(x,y)$ be twice continuously (real) differentiable on an open set. Suppose that the real and imaginary parts of $f(z)$ and $zf(z)$ are harmonic. Prove that f is analytic.

8. Let a be a complex number and suppose $|a| < 1$. Let $f(z) = \dfrac{z-a}{1-\bar{a}z}$. Prove the following statements.

   (a) $|f(z)| < 1$, if $|z| < 1$.

(b) $|f(z)| = 1$, if $|z| = 1$.

9. Suppose $P(z)$ is a polynomial and that all of its roots have positive real part. Prove that the zeros of the derivative of P have positive real part.

10. Let $z_j = e^{\frac{2\pi i j}{n}}$ denote the n roots of unity. Let $c_j = |1 - z_j|$ be the $n-1$ chord lengths from 1 to the points $z_j, j = 1, \dots . n-1$. Prove that the product $c_1 \cdot c_2 \cdots c_{n-1} = n$. *Hint*: Consider $z^n - 1$.

11. Let $f(z) = x^2 - y^2 + i\log(x^2 + y^2)$. Find the points at which f is complex differentiable. Find the points at which $g(z) = x - iy$ is complex analytic.

12. Let $f(z) = u(z) + iv(z), u = Re(f(z)), v = Imf((z))$ be analytic on an open connected set Ω. Suppose there are real numbers a, b, c with $a^2 + b^2 \neq 0$ and $au(z) + bv(z) = c$ for all $z \in \Omega$. Prove that f is constant.

13. Let f be analytic on $\{z : |z| > R\}$. Suppose $|f(z)| \leq |z|^k$. Let $f(z) = \sum_{-\infty}^{+\infty} a_n z^n$ be the Laurent expansion of f. Prove that $a_n = 0$ if $n > k$.

14. Let f be analytic within and on a simple closed curve Γ. Prove that $Re\left(\int_\Gamma \overline{f}(z) f'(z) dz\right) = 0$.

15. Compute $\displaystyle\int_{|z|=r} \frac{|dz|}{|z-a|^2}$, where $|a| \neq r$. Use the fact that on $\{|z| = r\},\ |dz| = -ir\dfrac{dz}{z}$; and then use the Cauchy integral formula.

16. You will need to know the definitions of the following terms and statements of the following theorems.

    (a) Modulus (absolute value) and argument of a complex number

    (b) Complex derivative

(c) Complex analytic function
(d) Cauchy-Riemann equations
(e) Harmonic functions and harmonic conjugate
(f) Complex exponential function
(g) Complex logarithm
(h) Cauchy's integral theorem and formula
(i) Maximum principle for harmonic functions
(j) Linear fractional transformations

13. There may be homework problems or example problems from the text on the midterm.