Cauchy Integral Formula

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Theorem 1. *Let f be analytic on an open set W and let $\{z : |z-a| \le r\} \subset W$. Let $|b-a| < r$. Then*

$$\int_{|z-a|=r} \frac{f(z)dz}{z-b} = 2\pi i f(b).$$

Proof. By the Cauchy integral theorem

$$\int_{|z-a|=r} \frac{f(z)-f(b)}{z-b} dz = 0.$$

Hence

$$\int_{|z-a|=r} \frac{f(z)dz}{z-b} = f(b) \int_{|z-a|=r} \frac{dz}{z-b}.$$

Now $\frac{1}{z-b}$ is infinitely complex differentiable when $|z-a| < r, |z-b > \epsilon$. Let $\frac{1}{z-b} = u + iv$. Apply Green's theorem to $\int_{\partial A}(u+iv)(dx+idy)$ where $A = \{z : |z-a| < r, |z-b > \epsilon\}$ to get

$$\int_A (-v_x - u_y) dxdy + \int_A (u_x - v_y) dxdy = 0$$

by the Cauchy-Riemann equations. Hence

$$\int_{|z-a|=r} \frac{f(z)dz}{z-b} = f(b) \int_{|z-a|=r} \frac{dz}{z-b} = f(b) \int_{|z-b|=\epsilon} \frac{dz}{z-b} = 2\pi i f(b).$$

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