Harmonic Functions on an Annulus

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Lemma 1. *Let* $p(z)_y = q(z)_x$ *on a convex set* W *in* $\mathbb{C}$*. Then* $v(z) = \int_{z_0}^{z} pdx + qdy$ *where* $z \in W, z_0 \in W$ *is well defined, can be computed along any curve from* z_0 *to* z *in* W*, and satisfies* $v(z)_x = p(z), v(z)_y = q(z)$*.*

Proof. This is a result we proved last quarter. □

Theorem 1. *Let* $p(z)_y = q(z)_x$ *on* $A = \{z : r < |z| < R\}$*. Suppose that* $\int_{|z|=a} pdx + qdy = 0$ *where* $r < a < R$*. Let* $v(z) = \int_C pdx + qdy$ *where* C *is the curve that consisting of the positively oriented arc of the circle of radius* a *from* a *to* $ae^{i\theta}$ *and then on the line segment from* $ae^{i\theta}$ *to* $z = be^{i\theta}$*. Then* v *is well defined and differentiable; and* $v_x = p, v_y = q$*.*

Proof. Because $\int_{|z|=a} pdx+qdy = 0$, we could also compute v by going along the circle $|z| = a$ in a clockwise direction. Let z_0 be a fixed point in A. By using the lemma we can compute $v(z)$ near z_0 by using the definition to compute $v(z_0)$ and then add the integral from z_0 to z along any curve in some convex set around z_0. Now we can prove $v_x(z_0) = p, v_y(z_0) = q$ by using the lemma. □

Corollary 1. *Let* u *be harmonic in* $A = \{z : r < |z| < R\}$*. If* $\int_{|z|=a} \frac{\partial u}{\partial n} ds = 0$ *then* u *has a harmonic conjugate* v *in* A *and* $u = \Re(f)$ *where* f *is analytic in* A*.*

Proof. let $p = -u_y, q = u_x$. Since u is harmonic, $p_y = q_x$ and

$$\int_{|z|=a} pdx + qdy = \int_{|z|=a} -u_y dx + u_x dy = \int_{|z|=a} \frac{\partial u}{\partial n} ds = 0.$$

By the theorem there is a function v such that $v_x = -u_y, v_y = u_x$ so u has a harmonic conjugate in A. □

Corollary 2. *Let* u *be a harmonic function in* $A = \{z : r < |z| < R\}$ *and let* $P = \int_{|z|=a} \frac{\partial u}{\partial n} ds$*. Then*

$$u - \frac{P}{2\pi} \log|z| = \Re(f),$$

where f *is analytic in* A*.*

Proof. Let $w = u - \frac{P}{2\pi} \log|z|$. Then w is harmonic and

$$\int_{|z|=a} \frac{\partial w}{\partial n} ds = P - \frac{P}{2\pi} 2\pi = 0.$$

□