Math 336 Sample Problems

One notebook sized page of notes will be allowed on the test. The test will cover up to §6.1 in the text (excluding those sections for which there was no homework).

1. Let f be an analytic function on an open connected set W. Suppose $0 \in W$ and suppose $|f(\frac{1}{n})| < e^{-n}$ for all $n > 0$. Prove that $f(z) = 0$ for all $z \in W$.

2. Using Rouché's theorem, show that $z^5 + 5z^3 + z - 2$ has three roots in the set $\{z : |z| < 1\}$.

3. Suppose $f(z)$ is an entire function and $|f(z)| < 1 + |z|^{1/2}$. Prove that f is constant.

4. Using contour integration Prove that
$$\int_0^\pi \frac{dt}{1 + \sin^2 t} = \frac{\pi}{\sqrt{2}}.$$

5. Let f be an entire function. Let $a \in \mathbb{C}, b \in \mathbb{C}$. Let $R > \max\{|a|, |b|\}$. Prove that
$$\frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz = \frac{f(b) - f(a)}{b - a} \text{ if } a \neq b$$

6. Suppose f is analytic on the square $Q = \{z : |x| < 1, |y| < 1\}$ and suppose f is continuous on the closure of Q. Denote the sides of the square by $S_j, j = 1, \ldots, 4$ starting with the rightmost side. Suppose $|f(z)| \leq R_j$ when $z \in S_j$. Prove
$$|f(0)|^4 \leq R_1 R_2 R_3 R_4.$$

7. Let $D = \{z : |z| < 1\}$. Suppose f is analytic on an open set that includes the closure of D and suppose $|f(z)| < 1$ if $|z| = 1$. Prove that there is a unique $\zeta \in D$ such that $f(\zeta) = \zeta$.

8. Let u and v be harmonic on an open connected set W. Suppose that $u(z)v(z) = 0$ on an open subset of W. Prove that either u or v is identically 0 on W.

9. Suppose f is analytic in $\{0 < |z| < r\}$ for some $r > 0$. Suppose also that $|f(z)| < |z|^{-1+\epsilon}$ in $\{0 < |z| < \delta\}$, where $\epsilon > 0$. Prove that f has a removable singularity at 0.

10. Let $D = \{z : |z| < 1\}$. Let f be analytic and non-constant on W, and suppose $\overline{D} \subset W$. Suppose $|f|$ is constant on ∂D. Prove that f has at least one zero in D.

11. Prove that
$$\sum_{n=1}^{\infty} d(n) z^n = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n} \text{ for} |z| < 1,$$
where $d(n)$ is the number of divisors of n. Carefully consider convergence issues.

12. Suppose f is entire and $|f(z)| \leq |Ke^z|$ for some K. Prove that $f(z) = Ce^z$ for some C.

13. There may be homework problems, example problems from the text, or requests for statements of theorems on the midterm.