Math 336 Sample Problems

One notebook sized page of notes (both sides) will be allowed on the test. The test will be comprehensive. The final exam is at 8:30 am, Monday, June 10, in the regular classroom.

1. Is there an analytic function f that maps $|z| < 1$ into $|z| < 1$ such that $f(\frac{1}{2}) = \frac{2}{3}, f(\frac{1}{4}) = \frac{1}{3}$?

2. Suppose u_n is a sequence of harmonic functions on a domain W and suppose the sequence converges uniformly on compact sets to a function u. Prove that u is harmonic.

3. Let $f(z) = \dfrac{z-a}{1-\bar{a}z}$, where $|a| < 1$. Let $D = \{z : |z| < 1\}$. Prove that

   (a)
$$\frac{1}{\pi}\int_D |f'(z)|^2 dxdy = 1.$$

   (b)
$$\frac{1}{\pi}\int_D |f'(z)| dxdy = \frac{1-|a|^2}{|a|^2}\log\left(\frac{1}{1-|a|^2}\right).$$

4. Let $u(x,y),\ v(x,y)$ be continuously differentiable as functions of (x,y) in a domain Ω. Let $f(z) = u(z) + iv(z)$. Suppose that for every $z_0 \in \Omega$ there is an r_0 (depending on z_0) such that
$$\int_{|z-z_0|=r} f(z)dz = 0,$$
   for all r with $r < r_0$. Prove that f is analytic in Ω. Hint: Show that f satisfies the Cauchy-Riemann equations in Ω.

5. Prove that all conformal maps from the upper half plane to the unit disk have the form
$$\alpha\frac{z-\beta}{z-\bar{\beta}},$$
   where $|\alpha| = 1, \Im(\beta) > 0$.

6. Let $D_2 = \{z : |z| < 2\}$ and $I = \{x \in \mathbf{R} : -1 \le x \le 1\}$. Find a bounded harmonic function u, defined in $D_2 - I$ such that u does not extend to a harmonic function defined in all of D_2.

7. Find a conformal map from the region between the two lines $y = x$ and $y = x + 2$ to the upper half plane, which sends 0 to 0.

8. Let f and g be entire functions so that satisfy $f^2 + g^2 = 1$. Prove that there is an entire function h so that $f = \cos(h), g = \sin(h)$.

9. Find a function, $h(x, y)$, harmonic in $\{x > 0,\ y > 0\}$, such that

$$h(x,y) = \begin{cases} 0 & \text{if } 0 < x < 2, y = 0, \\ 1 & \text{if } x > 2, y = 0, \\ 2 & \text{if } x = 0, y > 0 \end{cases}$$

10. Suppose that u is harmonic on all of $\mathbb{C}$ and $u \geq 0$. Prove that u is constant.

11. Suppose f is analytic on $H = \{z = x + iy : y > 0\}$ and suppose $|f(z)| \leq 1$ on H and $f(i) = 0$. Prove

$$|f(z)| \leq \left| \frac{z - i}{z + i} \right|.$$

12. Let $u(z)$ be harmonic in $\{z : 0 < |z| < 1\}$. Let $P = \displaystyle\int_{|z|=r} \frac{\partial u}{\partial n} ds$, where $0 < r < 1$. Show that P does not depend on r. Prove that

$$u(z) = \frac{P}{2\pi} \log|z| + \mathrm{Re}(f(z))$$

where f is analytic in $\{z : 0 < |z| < 1\}$.

13. Prove that if $|z| < 1$

$$\lim_{n\to\infty} \prod_{k=0}^{k=n} (1 + z^{2^k}) = \frac{1}{1 - z}$$

14. Suppose $f \in \mathcal{O}(0 < |z - a| < \epsilon)$ and that $\Re(f)$ is bounded. Prove that a is a removable singularity.

15. Let f be an analytic function defined on $\{|z| < 1\}$ such that $\Re(f(z)) \geq 0$.
    (a) Prove that $\Re(f(z)) > 0$.
    (b) Suppose $f(0) = 1$. Prove that

$$\frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|}.$$

16. Prove that $\displaystyle\sum_{1}^{\infty} \frac{\sin nz}{2^n}$ represents an analytic function on $|\Im(z)| < \log 2$.

17. Prove that $\sum_{1}^{\infty} \frac{e^{inz}}{n^2}$ represents an analytic function in $\Im(z) > 0$. Can you say anything about $\sum_{1}^{\infty} \frac{\sin nz}{n^2}$?

18. There may be homework problems or example problems from the text on the final. Don't forget previous sample problems.