Math 336 Sample Problems

One notebook sized page of notes (both sides) will be allowed on the test. The test will be comprehensive. The final exam is at 8:30 am, Monday, June 9, in the regular classroom.

1. Is there an analytic function f that maps $|z| < 1$ into $|z| < 1$ such that $f(\frac{1}{2}) = \frac{2}{3}, f(\frac{1}{4}) = \frac{1}{3}$?

2. Let $D = \{|z| < 1\}$. Suppose g is a real valued function on D and $0 \le g(z) \le |z|$. Suppose there is an $f \in \mathcal{O}(D)$ so that $|f(z)| = e^{g(z)}$. Prove that g is identically 0.

3. Suppose u_n is a sequence of harmonic functions on a domain W and suppose the sequence converges uniformly on compact sets to a function u. Prove that u is harmonic.

4. Let $f(z) = \dfrac{z-a}{1-\bar{a}z}$, where $|a| < 1$. Let $D = \{z : |z| < 1\}$. Prove that

   (a)
   $$\frac{1}{\pi}\int_D |f'(z)|^2 dxdy = 1.$$

   (b)
   $$\frac{1}{\pi}\int_D |f'(z)| dxdy = \frac{1-|a|^2}{|a|^2}\log\left(\frac{1}{1-|a|^2}\right).$$

   Hint: Use the Poisson integral formula.

5. Let $u(x, y)$, $v(x, y)$ be continuously differentiable as functions of (x, y) in a domain Ω. Let $f(z) = u(z) + iv(z)$. Suppose that for every $z_0 \in \Omega$ there is an r_0 (depending on z_0) such that
   $$\int_{|z-z_0|=r} f(z)dz = 0,$$
   for all r with $r < r_0$. Prove that f is analytic in Ω. Hint: Show that f satisfies the Cauchy-Riemann equations in Ω.

6. Suppose $u(x, y)$ is a harmonic function in a neighborhood of $|z| \le 1$ and suppose that u equals a polynomial $\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{m} a_{j,k}x^j y^k$ on $|z| = 1$. Prove that u is a polynomial.

7. Let $D_2 = \{z : |z| < 2\}$ and $I = \{x \in \mathbf{R} : -1 \le x \le 1\}$. Find a bounded harmonic function u, defined in $D_2 - I$ such that u does not extend to a harmonic function defined in all of D_2.

8. Suppose f is analytic on $D = \{|z| < 1\}$ and $f(0) = 0$. Prove that

$$\sum f(z^n)$$

converges uniformly on compact subsets of D.

9. Let a_k be a sequence of distinct complex numbers such that $\sum_{k=1}^{\infty} \frac{1}{|a_k|}$ converges.
Let $A = \{a_k : k = 1, \ldots, \infty\}$. Prove that

$$\sum_{k=1}^{\infty} \frac{1}{z - a_k}$$

converges to an analytic function on $\mathbb{C} - A$.

10. Let f and g be entire functions so that satisfy $f^2 + g^2 = 1$. Prove that there is an entire function h so that $f = \cos(h), g = \sin(h)$.

11. Find a function, $h(x, y)$, harmonic in $\{x > 0,\ y > 0\}$, such that

$$h(x,y) = \begin{cases} 0 & \text{if } 0 < x < 2, y = 0, \\ 1 & \text{if } x > 2, y = 0, \\ 2 & \text{if } x = 0, y > 0 \end{cases}$$

12. Suppose that u is harmonic on all of $\mathbb{C}$ and $u \geq 0$. Prove that u is constant.

13. Suppose f is analytic on $H = \{z = x + iy : y > 0\}$ and suppose $|f(z)| \leq 1$ on H and $f(i) = 0$. Prove

$$|f(z)| \leq \left| \frac{z - i}{z + i} \right|.$$

14. Compute

$$\int_0^{\infty} \frac{1 - \cos x}{x^2} dx.$$

15. Let f be a non-constant analytic function on the connected open set W. Let $Z = \{z : f(z) = 0\}$. Prove that $W - Z$ is connected.

16. (a) Prove that $1/z$ does not have an analytic antiderivative on $\mathbb{C} - \{0\}$.
    (b) Find all integers $0, \pm 1, \pm 2, \ldots$ such that the function $z^n e^{1/z}$ has an analytic antiderivative on $\mathbb{C} - \{0\}$.

17. Find the radius of convergence of

$$\sum \frac{n^n}{n!} z^{2n}.$$

18. Suppose $f \in \mathcal{O}(0 < |z-a| < \epsilon)$ and that $\operatorname{Re}(f)$ is bounded. Prove that a is a removable singularity.

19. Let f be a non-constant analytic function defined on $\{|z| < 1\}$ such that $\operatorname{Re}(f(z)) \geq 0$.
    (a) Prove that $\operatorname{Re}(f(z)) > 0$.
    (b) Suppose $f(0) = 1$. Prove that
    $$\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}.$$

20. Suppose f and g are analytic on a connected open set Ω.
    (a) If $|f(z)| + |g(z)|$ is constant, then both f and g are constant.
    (b) If $|f(z)| + |g(z)|$ assumes a local maximum in Ω, then f and g are constant.

21. Prove that $\displaystyle\sum_1^\infty \frac{\sin nz}{2^n}$ represents an analytic function on $|\operatorname{Im}(z)| < \log 2$.

22. Find all real valued harmonic functions on $\mathbb{C}$ that are constant on vertical lines (the constant may depend on the line).

23. Let f and g be two analytic functions on an open connected set W. Suppose that $f(z)\overline{g(z)}$ is real for all $z \in W$. Prove that either $f = cg$ or g is identically 0.

24. (a) Prove that the series
    $$\sum_1^\infty 2^{-n^2} z^{2^n}$$
    converges uniformly on $|z| \leq 1$.
    (b) Prove that the radius of convergence of the series is 1.

25. There may be homework problems or example problems from the text on the final. Don't forget previous sample problems.