Newton's Method

Newton's method works well in the complex case. This is a somewhat expanded version of my lecture.

Let f be analytic in a neighborhood of a point r which is a zero of f. Suppose that $f'(r) \neq 0$. Then there is a $\delta > 0$ so that if $|z_0 - r| < \delta$ then the iteration

$$z_{j+1} = z_j - \frac{f(z_j)}{f'(z_j)},$$

converges quadraticly (to be explained in the proof) to r.

Proof. We first derive a form of Taylor's formula for analytic functions. Recall

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = \rho} \frac{f(\zeta) d\zeta}{(\zeta - z)}.$$

Fix ρ and use the elephant-teacup formula in the form

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots + w^n + \frac{w^{n+1}}{1-w},$$

with

$$w = \frac{1}{1 - \left(\frac{z-a}{\zeta - a}\right)}.$$

Hence, with some manipulations, we get

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_n(z-a)^n + R(z,a)(z-a)^{n+1},$$

where

$$R(z,a) = \frac{1}{2\pi i} \int_{|\zeta - a| = \rho} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}(\zeta - z)}.$$

If we parametrize the circle $\zeta = a + \rho e^{i\theta}$ we see that the integrand is

$$\frac{f(a + \rho e^{i\theta}) i \rho e^{i\theta}}{\rho^{n+1} e^{i(n+1)\theta}(a + \rho e^{i\theta} - z)}.$$

Consider ρ as fixed and a, z as variables. Then the integrand depends continuously on a, z, θ so the function $R(z,a)$ is continuous (even analytic) in z, a.

We will apply this formula with $z = r, a = z_j$ to get

$$0 = f(r) = f(z_j) + f'(z_j)(r - z_j) + R(r, z_j)(r - z_j)^2.$$

Thus

$$\begin{aligned}
e_{j+1} = z_{j+1} - r = z_j - \frac{f(z_j)}{f'(z_j)} - r \\
&= z_j - r - \frac{1}{f'(z_j)}\left(f'(z_j)(z_j - r) - R(r, z_j)(z_j - r)^2\right) \\
&= \frac{R(r, z_j)}{f'(z_j)}(z_j - r)^2 \\
&= \frac{R(r, z_j)}{f'(z_j)} e_j^2.
\end{aligned}$$

Let $|\frac{R(r,z)}{f'(z)}| < C$ when $|z - r| < \epsilon$. Choose α so that $\alpha < \min\{1, \epsilon\}$ and so that $\delta = \frac{\alpha}{C} < \epsilon$. Then when $|z - r| < \min\{\delta, \epsilon\}$, $C|z - r| < \alpha < 1$.

The result of this is

$$|e_2| < C|e_1|^2 < C(C|e_0|^2)^2 = \frac{(C|z_0 - r|)^{2^2}}{C} < \frac{\alpha^{2^2}}{C}$$

and (by induction)

$$|e_j| < \frac{\alpha^{2^j}}{C}.$$

□