Residues

Note Title 4/28/2009

Suppose $f(z) = \dfrac{g(z)}{h(z)}$,

where g and h are analytic at z_0,

$g(z_0) \neq 0$, $h(z) = (z-z_0)^m H(z)$, $H(z_0) \neq 0$.

Let $H(z) = H_0 + H_1(z-z_0) + \cdots H_n(z-z_0)^n + \cdots$,

$g(z) = g_0 + g_1(z-z_0) + \cdots + g_n(z-z_0)^n + \cdots$,

$g_n = g^{(n)}(z_0)/n!$, $H_n = h^{(n+m)}(z_0)/(n+m)!$

Then $\operatorname{Res}[f, z_0] = \dfrac{1}{H_0^m} \det \begin{vmatrix} H_0 & 0 & 0 & \cdots & g_0 \\ H_1 & H_0 & & & g_1 \\ \vdots & & & & \\ H_{m-1} & \cdots & & H_1 & g_{m-1} \end{vmatrix}$

Proof:

$p_0 + p_1(z-z_0) + \cdots + p_{m-1}(z-z_0)^{m-1} + \cdots$

$= (z-z_0)^m f(z)$

$= g(z)/H(z) = \dfrac{\sum g_k (z-z_0)^k}{\sum H_k (z-z_0)^k}$, and

$f(z) = \dfrac{a_{-m}}{(z-z_0)^m} + \cdots + \dfrac{a_{-1}}{z-z_0} + \cdots$,

so $(z-z_0)^m f(z) = a_{-m} + \cdots + a_{-1}(z-z_0)^{m-1} + \cdots$.

Thus $a_{-1} = p_{m-1}$. We compute p_{m-1} by recurrence:

$$
\begin{aligned}
g_0 &= H_0 p_0 \\
g_1 &= H_1 p_0 + H_0 p_1 \\
&\vdots \\
g_{m-1} &= H_{m-1} p_0 + \cdots + H_0 p_{m-1},
\end{aligned}
\qquad
\begin{bmatrix} g_0 \\ \vdots \\ \vdots \\ g_{m-1} \end{bmatrix}
=
\begin{bmatrix}
H_0 & 0 & \cdots & & 0 \\
H_1 & H_0 & 0 & \cdots & 0 \\
H_2 & H_1 & H_0 & \cdots & 0 \\
\vdots & & & & \\
H_{m-1} & \cdots & \cdots & \cdots & H_0
\end{bmatrix}
\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{m-1} \end{bmatrix}
$$

Cramer's rule $\Rightarrow$

$$
a_{-1} = p_{m-1} = \frac{1}{H_0^m} \det \begin{bmatrix}
H_0 & 0 & \cdots & \cdots & g_0 \\
H_1 & H_0 & 0 & \cdots & g_1 \\
\vdots & & & & \\
H_{m-1} & \cdots & \cdots & H_1 & g_{m-1}
\end{bmatrix}
$$

If $m=2$ the residue is

$$
\left(\frac{2}{h''(z_0)}\right)^2 \left[g'(z_0)\frac{h''(z_0)}{2} - g(z_0)\frac{h'''(z_0)}{3!} \right]
$$

This result was taken from Hille, "Analytic Function Theory".
I don't know who should get the credit.