Maximum Principle

This is an improved (I hope) exposition of the one I gave in class.

Theorem 1. *Let u be harmonic on a bounded open set Ω. Suppose that*

$$\limsup_{z\to p} u(z) \le M,$$

for every $p \in \partial\Omega$. Then

$$u(z) \le M,$$

for all $z \in \Omega$.

Proof. Let $\delta > 0$. Let $W = \{z : u(z) > M + \delta\}$. Then

$$\partial W \subset \overline{\Omega}.$$

Then I claim $\partial W \cap \partial\Omega = \emptyset$. For if $p \in \partial W \cap \partial\Omega$, then $\limsup_{z\to p} u(z) \le M$, and $\limsup_{z\to p} u(z) \ge M + \delta$, which is a contradiction. So $\partial W \subset \Omega$, and hence $\overline{W} \subset \Omega$ and $\overline{W}$ is a compact set on which u is continuous. If $p \in \partial W$ then there must be nearby points which are in W and not in W. Thus there is a sequence of of points in $z_j \in \Omega$ with $z_j \to p$ with $u(z_j) \le M + \delta$ so the boundary values of u on ∂W are less than $M + \delta$. This implies that $u(z) \le M + \delta$ for $z \in W$. This is a contradiction. So $W = \emptyset$, and hence $u(z) \le M + \delta$ for all $z \in \Omega$. Since $\delta > 0$ is arbitrary, $u(z) \le M$ for all $z \in \Omega$. □