Poisson Integral Formula

June 2, 2014

This note contains a discussion of the Poisson Integral Formula. It will rely on some things we did in class this quarter and last quarter.

Let ϕ be Riemann integrable $[0, 2\pi]$. Define $u(z)$ for $|z| < 1$ by

$$u(z) = \frac{1}{2\pi}\mathrm{Re}\left(\int_0^{2\pi} \frac{(\zeta + z)}{(\zeta - z)}\phi(t)dt\right).$$

This expression can be written as a contour integral

$$u(z) = \frac{1}{2\pi}\mathrm{Re}\left(\int_0^{2\pi} \frac{(\zeta + z)}{(\zeta - z)}\frac{\phi(\zeta)}{i\zeta}d\zeta\right).$$

For most functions ϕ this integral cannot be evaluated explicitly. But, what if ϕ is identically 1? Then we can use the residue theorem. The integrand has simple poles at z and 0. The residues are $\frac{2}{i}, \frac{-1}{i}$. So $2\pi i$ times the sum of the residues is 2π. So in this case

$$u(z) = 1,$$

as we found in class by other means.