LU factorization

This note corrects errors I made in lecture on 10/1/08. The calculation proceeds as I described. The results are have the same meaning as in the lecture and are stored as described. Let me change the notation, keeping the m_{ij}'s as before but denoting the modified a_{ij}'s by u_{ij}. The result is:

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ m_{21} & u_{22} & \cdots & u_{2n} \\ m_{31} & m_{32} & u_{33} \cdots & u_{3n} \\ \vdots & & & \\ m_{n1} & \cdots & m_{n,n-1} & u_{nn} \end{bmatrix}$$

Let $L_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ m_{n1} & 0 & \cdots & 0 & 1 \end{bmatrix}$, $L_2 = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & m_{32} & 1 & 0 \cdots & 0 \\ 0 & m_{42} & 0 & 1 \cdots & 0 \\ \vdots & & & & \\ 0 & m_{n2} & 0 & \cdots & 1 \end{bmatrix}$, $\ldots$, $L_{n-1} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & 1 & 0 \\ 0 & & & m_{n,n-1} & 1 \end{bmatrix}$.

Let $L = L_1 \cdots L_{n-1}$. Then $L = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ l_{21} & 1 & \cdots & & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & \cdots & l_{n,n-1} & 1 \end{bmatrix}$ is lower triangular and

$$A = L \cdot U.$$

Now to solve $Ax = LUx = b$ we shouldn't compute L. The first step is to solve $Ly = b$. This can be written as

$$y = L^{-1}b = L_{n-1}^{-1} L_{n-2}^{-1} \cdots L_1^{-1} b; \text{ and}$$

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & - & - & 0 \\ -m_{31} & 0 & 1 & \cdot & \cdot & 0 \\ \vdots & & & & & \\ -m_{n1} & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}. \quad \text{Hence}$$

$$L_1^{-1} b = \begin{bmatrix} b_1 \\ b_2 - m_{21} b_1 \\ b_3 - m_{31} b_1 \\ \vdots \\ b_n - m_{n1} b_1 \end{bmatrix}. \quad \text{this is what we do.}$$

Don't compute L; just use the stored m_{ij}'s. Cost of solving $Ly = b$ is $(n-1) + (n-2) + \cdots + 1$ $= \frac{n(n-1)}{2}$ mult. In fact, it is easy to see that

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & 0 & \cdots & 0 \\ m_{31} & m_{32} & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ m_{n1} & m_{n2} & \cdots & & m_{n,n-1} & 1 \end{bmatrix}, \quad \text{i.e. } l_{ij} = m_{ij}$$