ith row. Thus the situation with the Jacobi method is similar to that of G[illegible] elimination in which the possibility of zero pivot elements must be guard[illegible] against.

Finally, we note that when D^{-1} exists, it is relatively easy (in comparison[illegible] a direct method) to carry out each step of the iteration. Thus in those cases [illegible] which $\|-D^{-1}(L + U)\| < 1$, the Jacobi method provides an alternative to di[illegible] methods.

Examination of (2.61) reveals that each component of the vector $x^{(k+1)}$ [illegible] computed entirely from the vector $x^{(k)}$. If $x_j^{(k+1)}$ is assumed to be closer to [illegible] true answer than $x_j^{(k)}$, the estimate for $x_i^{(k+1)}$ should be improved by repl[illegible] $x_j^{(k)}$ by $x_j^{(k+1)}$ whenever $j < i$. That is, we should use our most recent inform[illegible] as soon as it becomes available. The implementation of this idea leads to [illegible] procedure known as the Gauss-Seidel method.

If we use the new information as soon as it is available in (2.61), we o[illegible] (after multiplication by a_{ii}) this equation:

$$a_{ii}x_i^{(k+1)} = -\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} + b_i, \qquad i = 1, \ldots, n. \quad (2.[illegible])$$

(in which we interpret the first sum as zero when $i = 1$). We can write [illegible] equation in matrix form, using $A = L + D + U$ as in the Jacobi method, [illegible] obtain

$$D\mathbf{x}^{(k+1)} = -L\mathbf{x}^{(k+1)} - U\mathbf{x}^{(k)} + \mathbf{b}. \quad (2.[illegible])$$

Putting this in the standard form Eq. (2.56) for an iterative method, we [illegible]

$$(D + L)\mathbf{x}^{(k+1)} = -U\mathbf{x}^{(k)} + \mathbf{b}. \quad (2.[illegible])$$

The matrix $M_G = -(D + L)^{-1}U$ is called the *Gauss-Seidel matrix*. Since [illegible] Gauss-Seidel method is refinement of the Jacobi method, the former u[illegible] (but not always) converges faster. For deeper results on convergence and com[illegible] parison of rates of convergence, see the Ostrowski-Reich and Stein-Rosenb[illegible] Theorems in Varga (1962). Note that the choice of the starting vector $\mathbf{x}^{(0)}$ [illegible] particularly critical, and one natural choice is $\mathbf{x}^{(0)} = 0$. We will have more to [illegible] of this choice in Section 3.4.

EXAMPLE 2.16. As an example of the sorts of computational results that the Ja[illegible] and Gauss-Seidel methods give, consider the linear system

$$\begin{aligned} 3x_1 + x_2 + x_3 &= 5 \\ 2x_1 + 6x_2 + x_3 &= 9 \\ x_1 + x_2 + 4x_3 &= 6 \end{aligned} \qquad \text{with solution vector } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

With $\mathbf{x}^{(0)} = 0$, we obtain Tables 2.1 and 2.2. The coefficient matrix of the system [illegible] *diagonally dominant*, a condition that is sufficient to guarantee convergence o[illegible] Jacobi and Gauss-Seidel iterations (see Theorem 2.3).

TABLE 2.1 Jacobi iteration.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
1	0.166667E 01	0.150000E 01	0.150000E 01
2	0.666667E 00	0.694445E 00	0.708333E 00
3	0.119907E 01	0.115972E 01	0.115972E 01
4	0.893518E 00	0.907022E 00	0.910301E 00
5	0.106089E 01	0.105044E 01	0.104986E 01
6	0.966564E 00	0.971392E 00	0.972166E 00
7	0.101881E 01	0.101578E 01	0.101551E 01
8	0.989568E 00	0.991144E 00	0.991350E 00
9	0.100584E 01	0.100492E 01	0.100482E 01
10	0.996753E 00	0.997251E 00	0.997312E 00
11	0.100181E 01	0.100153E 01	0.100150E 01
12	0.998991E 00	0.999146E 00	0.999165E 00
13	0.100056E 01	0.100047E 01	0.100047E 01
14	0.999687E 00	0.999735E 00	0.999741E 00
15	0.100017E 01	0.100015E 01	0.100014E 01
16	0.999903E 00	0.999918E 00	0.999919E 00
17	0.100005E 01	0.100005E 01	0.100004E 01
18	0.999970E 00	0.999974E 00	0.999975E 00
19	0.100002E 01	0.100001E 01	0.100001E 01
20	0.999991E 00	0.999992E 00	0.999992E 00

TABLE 2.2 Gauss-Seidel iteration.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
1	0.166667E 01	0.944445E 00	0.847222E 00
2	0.106944E 01	0.100231E 01	0.982060E 00
3	0.100521E 01	0.100125E 01	0.998385E 00
4	0.100012E 01	0.100023E 01	0.999913E 00
5	0.999953E 00	0.100003E 01	0.100000E 01
6	0.999989E 00	0.100000E 01	0.100000E 01
7	0.999998E 00	0.100000E 01	0.100000E 01
8	0.100000E 01	0.100000E 01	0.100000E 01

As an example in which iteration is not so successful, consider the (4 × [illegible] system of Example 2.6 (solved by Gauss elimination in Example 2.[illegible] coefficient matrix is *positive-definite* and hence the Gauss-Seidel iterati[illegible] converges (see Theorem 2.4); but as can be seen, convergence is excee[illegible] [illegible] (see Table 2.3.) The question of how fast an iterative procedure w[illegible] [illegible] is considered in Section 3.4. Through the theory of the abov[illegible] [illegible] it can be shown that the Jacobi method will not converge f[illegible] [illegible] system above.