Van der Monde

Note Title 11/27/2009

One way to prove that there is a unique interpolating polynomial is to prove the formula

$$(1) \quad V(x_0, \cdots, x_n) = \det \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & & \cdots & \\ & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} = \prod_{i>j} (x_i - x_j)$$

The are generalizations of this idea to study interpolation with derivative data (Hermite interpolation). Here is a simple case. There is no need to prove this via determinants. It's just a neat statement.

HERMITE INTERPOLATION PROBLEM: Is there a (unique) polynomial p of degree $2n+1$ such that

$$p(x_0) = y_0, \; p'(x_0) = y_0', \; \cdots, \; p(x_n) = y_n, \; p'(x_n) = y_n' \quad ?$$

It's easy to see, using algebra, that the answer is yes. Here's an elegant proof using determinants.

The determinant of the matrix of this linear system is

$$
\det \begin{bmatrix}
1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^n \\
0 & 1 & 2x_0 & 3x_0^2 & \cdots & n x_0^{n-1} \\
1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^n \\
0 & 1 & 2x_1 & 3x_1^2 & \cdots & n x_1^{n-1} \\
\vdots & \vdots & & \vdots & & \vdots \\
1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^n \\
0 & 1 & 2x_n & 3x_n^2 & \cdots & n x_n^{n-1}
\end{bmatrix}
$$

This is equal to

$$
\left. \frac{\partial^{n+1} V(x_0, t_0, x_1, t_1, \ldots, x_n, t_n)}{\partial t_0 \, \partial t_1 \cdots \partial t_{n-1} \, \partial t_n} \right|_{(t_0, \ldots, t_n) = (x_0, \ldots, x_n)}
$$

This is the coefficient of $(t_0 - x_0)(t_1 - x_1) \cdots (t_n - x_n)$ in the Taylor series expansion of $V(x_0, t_0, \ldots, x_n, t_n)$ in powers of $(t_0 - x_0), (t_1 - x_1), \ldots, (t_n - x_n)$ — "based at $x_0, \ldots, x_n$". Using (1) we see this is

$$
\left(\prod_{j=0}^{n-1} \left[(t_n - t_j)(x_n - x_j) \right] \cdot \prod_{j=0}^{n-1} \left[(x_n - t_j)(x_n - x_j) \right] \right) \left(\prod_{j=0}^{n-2} \left[(t_{n-1} - t_j)(x_n - x_j) \right] \prod_{j=0}^{n-2} \left[(x_{n-1} - t_j)(x_{n-1} - x_j) \right] \right)
$$

$$
\times \cdots \; (t_2 - t_0)(x_2 - x_0)(t_2 - t_1)(x_2 - x_1) \times (x_2 - t_0)(x_2 - x_0)(x_2 - t_1)(x_2 - x_1)
$$
$$
\times \; (t_1 - t_0)(x_1 - x_0)(x_1 - t_0)(x_1 - x_0)
$$

evaluated at $(t_0, \ldots, t_n) = (x_0, x_1, \ldots, x_n)$.

This is $\prod_{j=0}^{n-1} (x_n - x_j)^4 \cdot \prod_{j=0}^{n-2} (x_{n-2} - x_j)^4 \cdots (x_2 - x_0)^4 (x_2 - x_1)^4 (x_1 - x_0)^4$

$$
= \left[V(x_0, \ldots, x_n) \right]^4 = \prod_{i > j} (x_i - x_j)^4 \neq 0.
$$

QED