# Fourier Analysis

February 21, 2011

This note is an exposition of some ideas in Fourier analysis. To make life simple, I'll deal only with continuous functions defined on $[0, 2\pi]$. I'll use the notation $e_j = e_j(x) = \exp(ijx)$. Here's the definition of an inner product on complex valued functions on $[0, 2\pi]$.

**Definition 1.**

$$< f, g >= \int_0^{2\pi} f(x)\bar{g}(x)dx, \ \|g\|^2 =< g, g > .$$

**Lemma 1.**

$$< e_j, e_k >= 2\pi\delta_j^k.$$

**Theorem 1.** *Let $f$ be a continuous function on $[0, 2\pi]$ and let $c_k = \frac{1}{2\pi} < f, e_k >$ . Let $f_N = \sum_{n=-N}^{N} c_n e_n$ and let $g = \sum_{-N}^{N} a_n e_n$, where $a_n$ are any complex numbers. Then*

$$\|f - f_N\|^2 \leq \|f - g\|^2.$$

*Hence $f_N$ is the best approximation to $f$ in the $\| \ \|$ sense among all functions that are linear combinations of $\{e_n\}_{-N}^{N}$.*

*Proof.*

$$< e_j, f - f_N >=< e_j, f > - < e_j, f_N >= 2\pi c_j - 2\pi c_j = 0.$$

Since $f_N - g$ is a linear combination of $e_j$, $< f - f_N, f_N - g >= 0$. Then we compute

$$\begin{aligned} \|f - g\|^2 &=< f - g, f - g >=< f - f_N + f_N - g, f - f_N + f_N - g > \\ &= \|f - f_N\|^2 + 2 < f - f_N, f_N - g > + \|f_N - g\|^2 \\ &= \|f - f_N\|^2 + \|f_N - g\|^2 \\ &\geq \|f - f_N\|^2. \end{aligned}$$

□

**Definition 2.** $\sum_{-\infty}^{\infty} c_n e_n$ *is the Fourier series of $f$. If $\lim_{N\to\infty} \sum_{-N}^{N} c_n e_n(x)$ converges we say the Fourier series converges. It may not converge and even if it converges it may not converge to $f(x)$.*

Now let's do the same thing discretely. The first thing is to approximate the integral $< f, e_k >= \int_0^{2\pi} f(x)\exp(-ikx)dx$ by a Riemann sum. Divide the interval $[0, 2\pi]$ into $n$ equal parts. Let $\delta = 2\pi/n, \ x_j = 2\pi j/n = j\delta$. Then a Riemann sum for the integral is

$$\frac{2\pi}{n} \sum_{j=1}^{n} \exp(\frac{-2\pi ijk}{n}) f_j,$$

where $f_j = f(x_j)$. Let $\omega = \exp(\frac{-2\pi i}{n})$. Then $\omega$ is a primitive $n^{th}$ root of unity and the approximation $\hat{f}_k$ to the coefficient $c_k$ takes the form

**Definition 3.**

$$\hat{f}_k = \frac{1}{n}\sum_{j=1}^{n}\omega^{kj}f_j.$$

Let $\Omega$ be the matrix defined by $\Omega_{i,j} = \omega^{ij}$, and let $f$ and $\hat{f}$ be column vectors with componets $f_j$ and $\hat{f}_j$. then Definitiion 3 can be written as matrix multiplication.

$$\hat{f} = \Omega f.$$

In this form Definition 3 is called the **Discrete Fourier Transform**. Let's find an analog of Lemma 1. First a few properties of $\Omega$.

**Lemma 2.**

$$\Omega^T = \Omega,\ \ \Omega\bar{\Omega} = nI,\ \ \Omega^{-1} = \frac{1}{n}\bar{\Omega}.$$

For this we need another lemma.

**Lemma 3.** *For any $n^{th}$ root of unity, $\mu \neq 1$,*

$$\sum_{k=1}^{n}\mu^k = 0.$$

*Proof.* (of Lemma 3) An $n^{th}$ root of unity satisfies $z^n - 1 = 0$. Hence $\mu^n - 1 = (\mu-1)(\mu^{n-1}+\mu^{n-2}+\cdots+\mu+1) = 0$. Since $\mu \neq 0$, $\mu^{n-1}+\mu^{n-2}+\cdots+\mu+1) = 0$. Multiply by $\mu$ to get $\sum_{k=1}^{n}\mu^k = 0$. ☐

*Proof.* (of Lemma 2)

The $k,\ell$ entry of $\Omega\bar{\Omega}$ is $\sum_{j=1}^{n}\omega^{kj}\omega^{-j\ell} = \sum_{j-1}^{n}(\omega^{k-l})^j = 0, k \neq \ell$. If $k = \ell$, the $k,k$ entry is $n$.

☐