Block Multiplication of Matrices

This note describes multiplication of block (partitioned matrices). A special case gives a representation of a matrix as a sum of rank one matrices. Suppose $[n] = (1, 2, \ldots, n)$ is the (ordered) sequence of integers from 1 to n. An *ordered partition* (my term) is a set of ordered subsets $J = (J_1, J_2, \ldots, J_p)$ which come from putting marks at $p - 1$ places in the ordered list $(1, 2, \ldots, n)$ at arbitrary locations, for example

$$J_1 = (1, 2, 3), J_2 = (4), J_3 = (5, 6)$$

is an ordered partition of [6]. The *size* of J is n and the *size* of J_k is the cardinality of f J_k. These are denoted by $|J|, |J_k|$.

Definition 1. *Let A be a matrix. Let $A(J_r; K_s)$, denote the submatrix of A with entries from row J_r and columns K_s. This defines a paritioning of A and we call A a partitioned matrix.*

Theorem 1. *Let J, K, L be ordered partitions of size p, q, r respectively. Let A, B and be partioned matrices with the partitions of A defined by J, K and the partitions of B defined by K, L. Let $C = AB$. Let the number of partitions in K be m. Then*

$$C(J_r; L_t) = \sum_{s=1}^{s=m} A(J_r; K_s)B(K_s; L_t).$$

Proof. The proof will be illustrated in a particular case. The argument is general and can be used to prove the theorem. Let take as an example a partition that includes the index $2 \in J_1$ and the index $4 \in L_2$. We want to identify the term c_{24} in the product matrix C. This entry is found by summing terms found by multiply entries in row 2 of A times corresponding entries in column 4 of B. This sum can be computed by adding the products of the entries in row 2 in blocks $A(J_1; K_s)$ times the entries in column 4 in the blocks $B(K_s; L_2)$. With an obvious simplified notation the theorem can be written

$$C_{rt} = \sum_s A_{rs}B_{st},$$

and the formula reads formally as if the entries A_{rs}, B_{st} are scalars.

□

Corollary 1. *Let $A = [A_1, A_2, \ldots, A_n], B = [B_1, B_2, \ldots, B_n]^T$ be matrices with n column vectors and n row vectors respectively. Then*

$$AB = A_1B_1^T + A_2B_2^T + \cdots + A_nB_n^T.$$