ELECTRICAL LIE THEORY

COLLIN LITTERELL

CONTENTS

1. Electrical Networks.

1.1. Preliminary Definitions.

A **graph with boundary** is a graph $G = (V, E)$ with a designated subset $\partial V \subset V$ such that every component of G has a vertex contained in ∂V. We write $\operatorname{Int} V = V \setminus \partial V$, and we call elements of ∂V **boundary vertices** and elements of $\operatorname{Int} V$ **interior vertices**. We will only deal with graphs that are finite, but we allow multiple edges and loops. We will use the notation $u \sim e \sim v$ to mean that u and v are (not necessarily distinct) vertices and e is an edge which is adjacent to both u and v. A **circular planar graph** is a graph with boundary which is embedded into the closed unit disk in the plane in such a way that each of the boundary vertices lies on the boundary of the disk and the rest of the graph lies within the interior of the disk. Whenever we number the boundary vertices of a circular planar graph, we will assume that they are arranged in clockwise order around the boundary of the disk. An **electrical network** is a pair (G, γ), where G is a graph with boundary and $\gamma : E \to \mathbb{R}_{>0}$ is a strictly positive function on the set of edges of G. We will typically denote an electrical network (G, γ) by Γ. After we introduce the electrical Lie algebra, we will restrict our attention to the case when Γ is a **circular planar electrical network**, meaning that the underlying boundary graph G is circular planar.

Let $\Gamma = (G, \gamma)$ be an electrical network, and label the vertices $v_1, \ldots, v_n$ in such a way that $v_1, \ldots, v_k \in \partial V$ and $v_{k+1}, \ldots, v_n \in \operatorname{Int} V$, where $k = |\partial V|$. We define the **Kirchhoff matrix** of Γ (with respect to this ordering of the vertices) to be the matrix

$$K_{ij} = \begin{cases} -\sum\limits_{e: v_i \sim e \sim v_j} (e) & \text{for } i \neq j, \\ \sum\limits_{\substack{e: v_i \sim e \sim v_k, \\ k \neq i}} (e) & \text{for } i = j. \end{cases}$$

For clarity, the first sum is over edges whose endpoints are v_i and v_j, and the second sum is over edges which have v_i as an endpoint and are not loops.

Remark. *K is symmetric.*

We can therefore write the Kirchhoff matrix K as

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where A is the $|\partial V| \times |\partial V|$ submatrix corresponding to the boundary vertices and C is the $|\operatorname{Int} V| \times |\operatorname{Int} V|$ submatrix corresponding to the interior vertices.

Remark. *The submatrix C of the Kirchhoff matrix that corresponds to the interior vertices is positive-definite. In particular, C is invertible.*

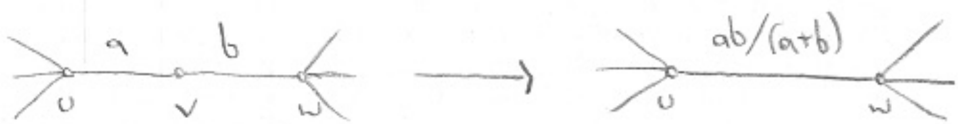
We now define the **response matrix** of Γ (with respect to this ordering of the vertices) to be the Schur complement of C in K, i.e., the matrix

$$\Lambda(\Gamma) = A - BC^{-1}B^T.$$

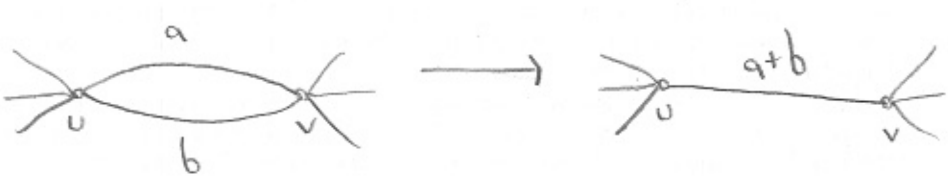
1.2. Network Transformations.

There are several simple transformations that we can apply make to an electrical network, as we will now discuss.

- Reducing a series connection: Suppose Γ has a pair of edges $u \sim e \sim v$ and $v \sim f \sim w$ with conductivities a and b, respectively, such that v is an interior node and has no other edges adjacent to it. We can obtain a new electrical network by removing v and replacing e and f with a single edge $u \sim g \sim w$ which has conductivity $ab/(a+b)$.



- Reducing a parallel connection: Suppose Γ has a pair of edges $u \sim e \sim v$ and $u \sim f \sim v$ with conductivities a and b, respectively. We can obtain a new electrical network by replacing e and f with a single edge g which has conductivity $a + b$.

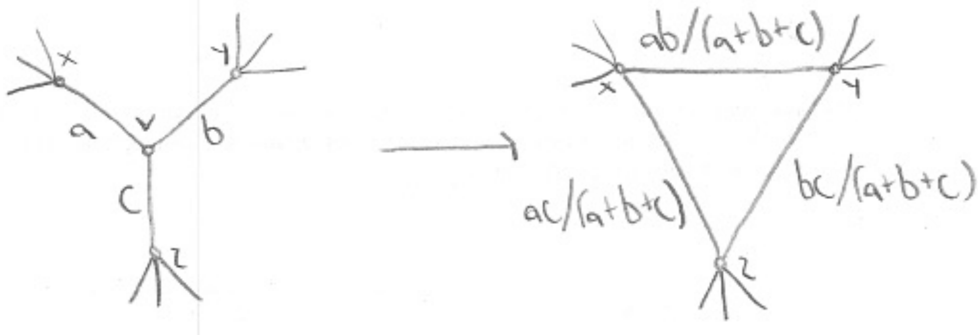


- Removing an interior spike: Suppose Γ has an interior vertex v of degree one. We can obtain a new electrical network by removing v and the adjacent edge.

- Removing a loop: Suppose Γ has a vertex with a loop. We can obtain a new electrical network by removing the loop.

Notice that each of the procedures described above can be performed in reverse. We will use the term **series transformation** to refer to either reducing or introducing a series connection, and the term **parallel transformation** to refer to either reducing or introducing a parallel connection.

Now we describe a transformation which replaces a "Y" in a circular planar electrical network Γ with a "Δ". Suppose Γ has vertices v, x, y, and z with edges $v \sim vx \sim x$, $v \sim vy \sim y$, and $v \sim vz \sim z$ such that v has no other adjacent edges. Denote the conductivities of the edges a, b, and c, respectively. We can obtain a new electrical network by removing v and replacing vx, vy, and vz with edges $x \sim xy \sim y$, $y \sim yz \sim z$, and $z \sim zx \sim x$ having conductivities $ab/(a+b+c)$, $bc/(a+b+c)$, and $ca/(a+b+c)$, respectively. We call this a $\mathbf{Y-\Delta}$ **transformation.** Notice that we can apply this procedure in reverse to transform a "Δ" into a "Y". We will refer to this type of transformation also as a $Y - \Delta$ transformation.



Proposition 1.1. *Series transformations, parallel transformations, and $Y - \Delta$ transformations do not change the response matrix for a network.*

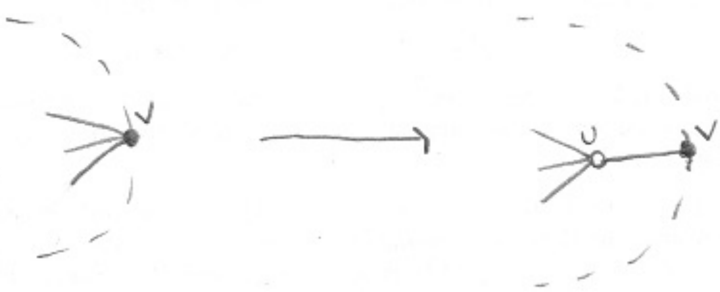
Let Γ be a circular planar electrical network, and label the boundary vertices $v_1, \ldots, v_n$ in a clockwise order. A **boundary edge** is an edge between two adjacent boundary vertices (where we consider v_n adjacent to v_1). A **boundary spike** is an edge between a boundary vertex of degree one and an interior vertex. We can transform an electrical network by removing any existing boundary edge or boundary spike, and we can also transform an electrical network by attaching a boundary edge or a boundary spike. More precisely, we have the following additional transformations that we can apply to a network.

- Removing a boundary edge: Suppose Γ has a boundary edge. We can obtain a new electrical network by simply removing the boundary edge.

- Contracting a boundary spike: Suppose Γ has a boundary spike e which connects a boundary vertex v and an interior vertex u. We can obtain a new electrical network by removing e and v and turning u into a boundary vertex which we will rename v.

- Attaching a boundary edge: Suppose v, w are adjacent boundary vertices in Γ. We can obtain a new electrical network by adding an edge between v and w with any specified conductivity.

- Attaching a boundary spike: Suppose v is a boundary vertex in Γ. We can obtain a new electrical network by turning v into an interior vertex and renaming it u, adding a new boundary vertex named v, and adding an edge between u and v with any specified conductivity.



Notice that any of the transformations that we have defined on electrical networks can be viewed as transformations of *boundary graphs* by simply ignoring any mention of conductivities.

2. Reduced Words and Braid Relations.

We will denote the **symmetric group** of degree n, which is the group of all permutations of the set $\{1, \ldots, n\}$, by S_n. For each $1 \leq i \leq n$, let σ_i denote the permutation in S_n which interchanges i and $i+1$ and fixes all other numbers. We call the permutations $\sigma_1, \ldots, \sigma_{n-1}$ in S_n the **adjacent transpositions**.

By a **word** over the set $\{1, \ldots, n\}$, we mean a finite sequence of elements of $\{1, \ldots, n\}$; formally speaking, a word over $\{1, \ldots, n\}$ is an element of the free monoid on $\{1, \ldots, n\}$. Given any word $i_1 \ldots i_k$ over $\{1, \ldots, n\}$, there is a corresponding permutation in S_n given by $\sigma_{i_1} \cdots \sigma_{i_k}$.

It is a fact that any permutation $w \in S_n$ can be expressed as a product of adjacent transpositions, and the smallest such number of adjacent transpositions is called the **length** of w. If $\sigma_{i_1} \cdots \sigma_{i_k}$ is such an expression for w of minimal length, then the word $\underline{i} = i_1 \cdots i_k$ is called a **reduced word** for w.

Observe that the adjacent transpositions satisfy the following relations, which we will call the **braid relations**:

(i) $\sigma_i^2 = 1$ for all i,
(ii) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| > 1$,
(iii) $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ for $|i-j| = 1$.

These relations are sometimes referred to as the "Coxeter relations" (for example in [Gar02]) since they are a special case of relations for a Coxeter group.

Now consider the following transformations that we can perform on words:

$$\cdots ij \cdots \longrightarrow \cdots ji \cdots \quad \text{for } |i-j| > 1,$$
$$\cdots iji \cdots \longrightarrow \cdots jij \cdots \quad \text{for } |i-j| = 1.$$

We will refer to these transformations as **braid moves**. Observe that a word $\underline{i}$ can be obtained by applying a sequence of braid moves to another word $\underline{j}$ if and only if $\underline{j}$ can be obtained by applying a sequence of braid moves $\underline{i}$; in this case we say that $\underline{i}$ and $\underline{j}$ are related by a sequence of braid moves. The braid moves then give rise to an equivalence relation on words in the following way: $\underline{i} \sim \underline{j}$ if and only if $\underline{i}$ and $\underline{j}$ are related by a sequence of braid moves.

It is easy to see that the braid moves correspond in some way to the braid relations (ii) and (iii) from above; this will be made precise in the following Proposition. But first, notice that there is another transformation performed on words which corresponds to the braid relation (i), called a **nil-move**:

$$\cdots ijjk \cdots \longrightarrow \cdots ik \cdots \quad \text{for all } i, j, k.$$

Proposition 2.1. *If two words $\underline{i} = i_1 \cdots i_k$ and $\underline{j} = j_1 \cdots j_\ell$ are related by a sequence of braid moves, then they determine the same permutation, i.e., $\sigma_{i_1} \cdots \sigma_{i_k} = \sigma_{j_1} \cdots \sigma_{j_\ell}$.*

Proof. First suppose that $\underline{i}$ is obtained by performing a single braid move to $\underline{j}$. Then by applying the corresponding braid relation to the expression $\sigma_{i_1} \cdots \sigma_{i_k}$, we get the expression $\sigma_{j_1} \cdots \sigma_{j_\ell}$. The general result follows by a simple induction. □

The converse of Proposition 2.1 is also true (it was first proved by Jacques Tits in [Tit69]). We state this and another result here; proofs can be found in [Gar02] and [BB05].

Theorem 2.2. *(i) Any two reduced words for the same permutation are related by a sequence of braid moves.*
(ii) Let $\sigma_{j_1} \cdots \sigma_{j_\ell}$ be any expression for a permutation w and write $\underline{j} = j_1 \cdots j_\ell$. Then there is a reduced word $\underline{i}$ for w which can be obtained by applying a sequence of braid moves and nil-moves to $\underline{j}$.

3. Lie Groups and Lie Algebras.

This section will be an extremely brief introduction to Lie groups and Lie algebras. We will state only the definitions and results which will be important to us once we start talking about the electrical Lie algebra and the electrical Lie group.

3.1. Lie Groups.

A **Lie group** is a group that is also a smooth manifold, such that the multiplication map $G \times G \to G$ is smooth. One can think of a smooth manifold as a generalization of a surface. Then, as stated in [Hal15], a Lie group can be thought of as a "continuous group." Let us look at some simple examples of Lie groups.

Examples. *(i) One of the simplest examples of a Lie group is $\mathbb{R}^n$, where the "multiplication" is given by addition.*
(ii) The **general linear group** $\mathrm{GL}_n(\mathbb{R})$, *which is the group of $n \times n$ invertible real matrices, is a Lie group whose multiplication is given by matrix multiplication.*
(iii) The **special linear group** $\mathrm{SL}_n(\mathbb{R})$, *which is the group of $n \times n$ real matrices with determinant 1, is also a Lie group.*
(iv) Consider the $2n \times 2n$ matrix

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I denotes the $n \times n$ identity matrix. The **real symplectic group** *is the group*

$$\mathrm{Sp}_{2n} = \{2n \times 2n \text{ real matrices } A \text{ satisfying } A^T \Omega A = \Omega\},$$

which is also a Lie group. The real symplectic group is especially important to us because it is closely related to (in fact, it is isomorphic to) the electrical Lie group, which we will introduce in Section 5.

If G and H are Lie groups, a **Lie group homomorphism** from G to H is a group homomorphism $f : G \to H$ that is also a smooth map. A Lie group homomorphism $f : G \to H$ is a **Lie group isomorphism** if there exists a Lie group homomorphism $f^{-1} : H \to G$ such that $f^{-1} \circ f = \mathrm{id}_G$ and $f \circ f^{-1} = \mathrm{id}_H$.

3.2. Lie Algebras.

Let F be a field. A **Lie algebra** over F is a vector space $\mathfrak{g}$ over F equipped with a map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following properties:

(i) *Bilinearity.* For all $X, Y, Z \in \mathfrak{g}$ and $a, b \in F$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[X, aY + bZ] = a[X, Y] + b[X, Z].$$

(ii) *Antisymmetry.* For all $X, Y \in \mathfrak{g}$,

$$[X, Y] = -[Y, X].$$

(iii) *Jacobi identity.* For all $X, Y, Z \in \mathfrak{g}$,

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

We call the map $[-,-]$ the bracket on $\mathfrak{g}$. We will restrict our attention to Lie algebras over $\mathbb{R}$, which we refer to as real Lie algebras.

Examples. *(i) The space of all $n \times n$ real matrices form a real Lie algebra with bracket given by $[A, B] = AB - BA$. This Lie algebra is denoted $\mathfrak{gl}_n$.*

(ii) Consider the space $\mathfrak{sl}_n = \{A \in \mathfrak{gl}_n : \operatorname{tr} A = 0\}$. Recall that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any real $n \times n$ matrices A and B, therefore $\mathfrak{sl}_n$ is closed under the bracket operation $[A, B] = AB = BA$. Thus, $\mathfrak{sl}_n$ is also a Lie algebra.

(iii) Let $\mathfrak{sp}_{2n} = \{A \in \mathfrak{gl}_{2n} : A^T\Omega + \Omega A = 0\}$, where Ω is again the $2n \times 2n$ matrix

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Notice that for any $A, B \in \mathfrak{sp}_{2n}$,

$$\begin{aligned}(AB - BA)^T\Omega + \Omega(AB - BA) &= B^TA^T\Omega - A^TB^T\Omega + \Omega AB - \Omega BA \\ &= B^T(A^T\Omega + \Omega A) - (B^T\Omega + \Omega B)A \\ &\quad - A^T(B^T\Omega + \Omega B) + (A^T\Omega + \Omega A)B \\ &= 0,\end{aligned}$$

so $\mathfrak{sp}_{2n}$ is a real Lie algebra with bracket also given by $[A, B] = AB - BA$. Observe that for any $n \times n$ real matrices A, B, C, and D,

$$\begin{aligned}\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} -C^T & A^T \\ -D^T & B^T \end{pmatrix} + \begin{pmatrix} C & D \\ -A & -B \end{pmatrix} \\ &= \begin{pmatrix} C - C^T & A^T + D \\ -A - D^T & B^T - B \end{pmatrix},\end{aligned}$$

So, a block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is in $\mathfrak{sp}_{2n}$ if and only if $A^T = -D$, $B^T = B$, and $C^T = C$.

(iv) Any associative algebra A can be given a Lie algebra structure by defining the bracket to be the commutator bracket, $[x, y] = xy - yx$.

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a **Lie algebra homomorphism** is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ of vector spaces such that $\phi([X,Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$. A Lie algebra homomorphism ϕ is a **Lie algebra isomorphism** if there exists a Lie algebra homomorphism $\phi^{-1} : \mathfrak{h} \to \mathfrak{g}$ such that $\phi^{-1} \circ \phi = \mathrm{id}_{\mathfrak{g}}$ and $\phi \circ \phi^{-1} = \mathrm{id}_{\mathfrak{h}}$. Observe that if a Lie algebra homomorphism is bijective, then its inverse must also be a Lie algebra homomorphism. Therefore, a Lie algebra homomorphism is an isomorphism if and only if it is bijective.

If $\mathfrak{g}$ is a Lie algebra, a **Lie subalgebra** of $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ which is closed under the bracket operation inherited from $\mathfrak{g}$, i.e., $[X,Y] \in \mathfrak{h}$ whenever $X, Y \in \mathfrak{h}$. An **ideal** of $\mathfrak{g}$ is a subspace $\mathfrak{k}$ of $\mathfrak{g}$ such that $[X,Y] \in \mathfrak{k}$ whenever $X \in \mathfrak{k}$ and $Y \in \mathfrak{g}$. Given a subset X of $\mathfrak{g}$, the **Lie subalgebra (ideal) generated by** X is the smallest Lie subalgebra (ideal) of $\mathfrak{g}$ containing X. If $\mathfrak{k}$ is an ideal in $\mathfrak{g}$, then we can define the **quotient Lie algebra** to be the subspace $\mathfrak{g}/\mathfrak{k}$ (as a vector space quotient) with bracket given by $[X + \mathfrak{k}, Y + \mathfrak{k}] = [X,Y] + \mathfrak{k}$ (one can check that this is well-defined since $\mathfrak{k}$ is an ideal).

Later we will want to construct a Lie algebra in terms of generators and relations, much like we might do for a group. The following description of how this is done comes from [Hum78]. Let X be a set, and take V to be the real vector space generated by X. Consider the tensor algebra

$$T(V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

Since $T(V)$ is an associative algebra, the commutator bracket gives it a Lie algebra structure. Let L be the Lie subgroup of $T(V)$ generated by X. We call L the **free Lie algebra generated by** X. If we are given a subset $R \subset L$, the **Lie algebra with generators** X **and relations** R is defined to be the Lie algebra L/I, where I is the ideal of L generated by R.

Now if we are given a Lie group G, it will give rise to a Lie algebra $\mathrm{Lie}(G)$. Moreover, if we are given a Lie group homomorphism, it will give rise to a Lie algebra homomorphism $\Phi_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$. In fact, the assignment $G \mapsto \mathrm{Lie}(G)$, $\Phi \mapsto \Phi_*$ is a covariant functor from the category of Lie groups to the category of finite-dimensional real Lie algebras.

Remark. *The Lie algebras of* $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, *and* Sp_{2n} *are* $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, *and* $\mathfrak{sp}_{2n}$, *respectively.*

3.3. The Exponential Map.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. There is a smooth map $\exp : \mathfrak{g} \to G$, called the **exponential map**. While we won't actually give an explicit definition of the exponential map, we will state some of its properties. The following results be found in [Lee13].

Proposition 3.1. *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$*.properties.*

(i) For any $X \in \mathfrak{g}$ *and* $s, t \in \mathbb{R}$, $\exp((s+t)X) = \exp(sX)\exp(tX)$.

(ii) If G *is connected, then for any* $X, Y \in \mathfrak{g}$ *we have* $[X,Y] = 0$ *if and only if*

$$\exp(tX)\exp(sY) = \exp(sY)\exp(tX)$$

for all $s, t \in \mathbb{R}$.

(iii) If H *is another Lie group with Lie algebra* H *and* $\Phi : G \to H$ *is a Lie group homomorphism, then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\Phi_*} & \mathfrak{h} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow[\Phi]{} & H. \end{array}$$

Remark. *If G is any Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ (e.g., Sp_{2n} is a subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$) with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n$, then the exponential map $\exp : G \to \mathfrak{g}$ is given by*

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2}A^2 + \cdots .$$

Theorem 3.2. *(Lie Correspondence Theorem) The covariant functor given by the assignment $G \mapsto \mathrm{Lie}(G)$, $\Phi \mapsto \Phi_*$ is an equivalence of categories between the category of simply connected Lie groups and the category of finite-dimensional Lie algebras.*

4. The Electrical Lie Algebra $\mathfrak{el}_{2n}$.

Define the **electrical Lie algebra**, denoted $\mathfrak{el}_{2n}$, to be the real Lie algebra generated by $e_1, \dots, e_{2n}$ subject to the relations

$$[e_i, e_j] = 0 \text{ if } |i - j| > 1$$
$$[e_i, [e_i, e_j]] = -2e_i \text{ if } |i - j| = 1.$$

The electrical Lie algebra was introduced in [LP15]. The motivation behind the relations used to define the electrical Lie algebra is somewhat mysterious, but the important fact is that once we use this to define the electrical Lie group, we get relations that suggest some sort of relation to electrical networks (Proposition 5.1). It should also be mentioned that one advantage of this definition is that it allows us to generalize $\mathfrak{el}_{2n}$ to electrical Lie algebras of finite Dynkin types; see [Su14] for a discussion on this.

Let $\varepsilon_1, \dots, \varepsilon_n$ denote the standard basis vectors in $\mathbb{R}^n$. Let

$$a_1 = \varepsilon_1,\ a_2 = \varepsilon_1 + \varepsilon_2,\ a_3 = \varepsilon_2 + \varepsilon_3, \dots, a_n = \varepsilon_{n-1} + \varepsilon_n,$$
$$b_1 = \varepsilon_1,\ b_2 = \varepsilon_2, \dots, b_n = \varepsilon_n,$$

and for each $1 \le i \le n$, let $A_i = a_i\, a_i^T$ and $B_i = b_i\, b_i^T$. Note that

$$a_k^T\, b_\ell = \begin{cases} 1, & k = \ell \text{ or } k - 1 = \ell \\ 0, & \text{otherwise,} \end{cases}$$

therefore we have the following equalities:

$$A_k B_\ell = a_k\,(a_k^T\, b_\ell)\, b_\ell^T = \begin{cases} a_k\, b_\ell^T, & k = \ell \text{ or } k-1 = \ell \\ 0, & \text{otherwise,} \end{cases}$$

$$B_\ell A_k = b_\ell\,(b_\ell^T\, a_k)\, a_k^T = \begin{cases} b_\ell\, a_k^T, & k = \ell \text{ or } k-1 = \ell \\ 0, & \text{otherwise,} \end{cases}$$

$$A_k B_\ell A_k = a_k\,(a_k^T\, b_\ell)\,(b_\ell^T\, a_k)\, a_k^T = \begin{cases} A_k & k = \ell \text{ or } k-1 = \ell \\ 0, & \text{otherwise,} \end{cases}$$

$$B_\ell A_k B_\ell = b_\ell\,(b_\ell^T\, a_k)\,(a_k^T\, b_\ell)\, b_\ell^T = \begin{cases} B_\ell & k = \ell \text{ or } k-1 = \ell \\ 0, & \text{otherwise.} \end{cases}$$

The following result is from [LP15].

Theorem 4.1. *$\mathfrak{el}_{2n}$ and $\mathfrak{sp}_{2n}$ are isomorphic as Lie algebras.*

Proof. Define a Lie algebra homomorphism $\phi : \mathfrak{el}_{2n} \to \mathfrak{sp}_{2n}$ by

$$\phi(e_{2i-1}) = \begin{pmatrix} 0 & A_i \\ 0 & 0 \end{pmatrix},\ \phi(e_{2i}) = \begin{pmatrix} 0 & 0 \\ B_i & 0 \end{pmatrix}.$$

In order to check that ϕ is well-defined, we must show that it preserves the relations in $\mathfrak{el}_{2n}$; that is, we must show that the relations in $\mathfrak{el}_{2n}$ above still hold in $\mathfrak{sp}_{2n}$ when we replace each e_i with $\phi(e_i)$.

Let us compute $[\phi(e_i), \phi(e_j)]$ for different cases of i and j:

$$\begin{aligned}
[\phi(e_{2k-1}), \phi(e_{2\ell-1})] &= \left[\begin{pmatrix} 0 & A_k \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & A_\ell \\ 0 & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & A_k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & A_\ell \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & A_\ell \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & A_k \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \\
[\phi(e_{2k}), \phi(e_{2\ell})] &= \left[\begin{pmatrix} 0 & 0 \\ B_k & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ B_\ell & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & 0 \\ B_k & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B_\ell & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ B_\ell & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B_k & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \\
[\phi(e_{2k-1}), \phi(e_{2\ell})] &= \left[\begin{pmatrix} 0 & A_k \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ B_\ell & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & A_k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B_\ell & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ B_\ell & 0 \end{pmatrix}\begin{pmatrix} 0 & A_k \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} A_k B_\ell & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & B_\ell A_k \end{pmatrix} \\
&= \begin{pmatrix} A_k B_\ell & 0 \\ 0 & -B_\ell A_k \end{pmatrix}; \\
[\phi(e_{2k}), \phi(e_{2\ell-1})] &= -[\phi(e_{2\ell-1}), \phi(e_{2k})] \\
&= -\begin{pmatrix} A_\ell B_k & 0 \\ 0 & -B_k A_\ell \end{pmatrix} \\
&= \begin{pmatrix} -A_\ell B_k & 0 \\ 0 & B_k A_\ell \end{pmatrix}.
\end{aligned}$$

We wish to show that $[\phi(e_i), \phi(e_j)] = 0$ for $|i-j| > 1$, which we have already shown in the case that i and j have the same parity. If i is odd and j is even, write $i = 2k-1$ and $j = 2\ell$. Then $k \neq \ell$ and $k - 1 \neq \ell$ since $|i-j| > 1$, so $A_k B_\ell = 0 = B_\ell A_k$ and hence

$$[\phi(e_i), \phi(e_j)] = \begin{pmatrix} A_k B_\ell & 0 \\ 0 & -B_\ell A_k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In the case that i is even and j is odd, we have

$$[\phi(e_i), \phi(e_j)] = -[\phi(e_j), \phi(e_i)] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

by the antisymmetry of the bracket and the previous case.

Now we want to show that $[\phi(e_i), [\phi(e_i), \phi(e_j)]] = -2\phi(e_i)$ for $|i-j| = 1$. First suppose that i is odd. If we write $i = 2k-1$ and $j = 2\ell$, then we must either have $k = \ell$ or $k - 1 = \ell$ since $|i - j| = 1$. Therefore,

$$\begin{aligned}
[\phi(e_i), [\phi(e_i), \phi(e_j)]] &= [\phi(e_{2k-1}), [\phi(e_{2k-1}), \phi(e_{2\ell})]] \\
&= \begin{pmatrix} 0 & A_k \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_k B_\ell & 0 \\ 0 & -B_\ell A_k \end{pmatrix} - \begin{pmatrix} A_k B_\ell & 0 \\ 0 & -B_\ell A_k \end{pmatrix} \begin{pmatrix} 0 & A_k \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -A_k B_\ell A_k \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & A_k B_\ell A_k \\ 0 & 0 \end{pmatrix} \\
&= -2 \begin{pmatrix} 0 & A_k \\ 0 & 0 \end{pmatrix} \\
&= -2\phi(e_i).
\end{aligned}$$

Now suppose that i is even, and write $i = 2k$ and $j = 2\ell - 1$. Then either $\ell = k$ or $\ell - 1 = k$ since $|i - j| = 1$, so

$$\begin{aligned}
[\phi(e_i), [\phi(e_i), \phi(e_j)]] &= [\phi(e_{2k}), [\phi(e_{2k}), \phi(e_{2\ell-1})]] \\
&= \begin{pmatrix} 0 & 0 \\ B_k & 0 \end{pmatrix} \begin{pmatrix} -A_\ell B_k & 0 \\ 0 & B_k A_\ell \end{pmatrix} - \begin{pmatrix} -A_\ell B_k & 0 \\ 0 & B_k A_\ell \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B_k & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ -B_k A_\ell B_k & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ B_k A_\ell B_k & 0 \end{pmatrix} \\
&= -2 \begin{pmatrix} 0 & 0 \\ B_k & 0 \end{pmatrix} \\
&= -2\phi(e_i).
\end{aligned}$$

Thus, we have shown that $\phi : \mathfrak{el}_{2n} \to \mathfrak{sp}_{2n}$ is a well-defined Lie algebra homomorphism. A proof that ϕ is bijective is given in [LP15].

□

5. The Electrical Lie Group EL_{2n}.

By the Lie Correspondence Theorem, there exists a unique (up to Lie group isomorphism) simply connected Lie group EL_{2n} whose Lie algebra is $\mathfrak{el}_{2n}$. We call EL_{2n} the **electrical Lie group**.

Note that all of the results of this section come from [LP15].

Remark. *Since $\mathfrak{el}_{2n} \cong \mathfrak{sp}_{2n}$ as Lie algebras, $\mathrm{EL}_{2n} \cong \mathrm{Sp}_{2n}$ as Lie groups by the Lie correspondence theorem.*

For each $i = 1, \ldots, 2n$, consider the one-parameter subgroups $u_i : \mathbb{R} \to \mathrm{EL}_{2n}$ defined by $u_i(t) = \exp(te_i)$.

Proposition 5.1. *The elements $u_i(t) = \exp(te_i)$ of EL_{2n} satisfy the following relations*

(i) $u_i(s)u_i(t) = u_i(st)$,
(ii) $u_i(s)u_j(t) = u_j(s)u_i(t)$ *if* $|i-j| > 1$,
(iii) $u_i(r)u_j(s)u_i(t) = u_j(st/(r+t+rst))u_i(r+t+rst)u_j(rs/(r+t+rst))$ *if* $|i-j| = 1$.

Proof. (i) By part (i) of Proposition 3.1,

$$u_i(s)u_i(t) = \exp(se_i)\exp(te_i) = \exp(se_i + te_i) = \exp((s+t)e_i) = u_i(s+t).$$

(ii) Since $[e_i, e_j] = 0$ for $|i-j| > 1$, by part (ii) of Proposition 3.1 we have

$$u_i(s)u_j(t) = \exp(se_i)\exp(te_j) = \exp(te_j)\exp(se_i) = u_j(t)u_i(s).$$

(iii) By the Lie Correspondence Theorem and part (iii) of Proposition 3.1, since $\phi : \mathfrak{el}_{2n} \to \mathfrak{sp}_{2n}$ is a Lie algebra isomorphism, there exists a Lie group isomomorphism $\Phi : \mathrm{EL}_{2n} \to \mathrm{Sp}_{2n}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{el}_{2n} & \xrightarrow{\phi} & \mathfrak{sp}_{2n} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ \mathrm{EL}_{2n} & \underset{\Phi}{\dashrightarrow} & \mathrm{Sp}_{2n} . \end{array}$$

Then $u_i(t) = \exp(te_i) = \Phi^{-1}(\exp(\phi(te_i)))$. Therefore, since Φ is an isomorphism, it suffices to check the relation when we replace each $u_i(t)$ with $\exp(\phi(te_i))$. Notice that

$$\begin{aligned}
\exp(\phi(te_{2k-1})) &= \exp\begin{pmatrix} 0 & tA_k \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & tA_k \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & tA_k \\ 0 & 0 \end{pmatrix}^2 + \cdots \\
&= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & tA_k \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \cdots \\
&= \begin{pmatrix} I & tA_k \\ 0 & I \end{pmatrix} \\
\exp(\phi(te_{2\ell})) &= \exp\begin{pmatrix} 0 & 0 \\ tB_\ell & 0 \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ tB_\ell & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ tB_\ell & 0 \end{pmatrix}^2 + \cdots \\
&= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ tB_\ell & 0 \end{pmatrix} + 0 + \cdots \\
&= \begin{pmatrix} I & 0 \\ tB_\ell & I \end{pmatrix}.
\end{aligned}$$

If i is odd and $|i - j| = 1$, then

$$\begin{aligned}
\exp(\phi(re_i))\exp(\phi(se_j))\exp(\phi(te_i)) &= \exp(\phi(re_{2k-1}))\exp(\phi(se_{2\ell}))\exp(\phi(te_{2k-1})) \\
&= \begin{pmatrix} I & rA_k \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ sB_\ell & I \end{pmatrix}\begin{pmatrix} I & tA_k \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} I + rsA_kB_\ell & rA_k \\ sB_\ell & I \end{pmatrix}\begin{pmatrix} I & tA_k \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} I + rsA_kB_\ell & tA_k + rstA_kB_\ell A_k + rA_k \\ sB_\ell & stB_\ell A_k + I \end{pmatrix} \\
&= \begin{pmatrix} I + rsA_kB_\ell & (r + t + rst)A_k \\ sB_\ell & stB_\ell A_k + I \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&\exp(\phi(st/(r+t+rst)e_j))\exp(\phi((r+t+rst)e_i))\exp(\phi(rs/(r+t+rst)e_j))\\
&\quad= \exp(\phi(st/(r+t+rst)e_{2\ell}))\exp(\phi((r+t+rst)e_{2k-1}))\exp(\phi(rs/(r+t+rst)e_{2\ell}))\\
&\quad= \begin{pmatrix} I & 0 \\ st/(r+t+rst)B_\ell & I\end{pmatrix}\begin{pmatrix} I & (r+t+rst)A_k \\ 0 & I\end{pmatrix}\begin{pmatrix} I & 0 \\ rs/(r+t+rst)B_\ell & I\end{pmatrix}\\
&\quad= \begin{pmatrix} I & (r+t+rst)A_k \\ st/(r+t+rst)B_\ell & stB_\ell A_k + I\end{pmatrix}\begin{pmatrix} I & 0 \\ rs/(r+t+rst)B_\ell & I\end{pmatrix}\\
&\quad= \begin{pmatrix} I + rsA_kB_\ell & (r+t+rst)A_k \\ (st+rs)/(r+t+rst)B_\ell + rs^2t/(r+t+rst)B_\ell A_k B_\ell & stB_\ell A_k + I\end{pmatrix}\\
&\quad= \begin{pmatrix} I + rsA_kB_\ell & (r+t+rst)A_k \\ s(r+t+rst)/(r+t+rst)B_\ell & stB_\ell A_k + I\end{pmatrix}\\
&\quad= \begin{pmatrix} I + rsA_kB_\ell & (r+t+rst)A_k \\ sB_\ell & stB_\ell A_k + I\end{pmatrix}.
\end{aligned}$$

If i is instead even and $|i-j|=1$, let

$$\alpha = st/(r+t+rst),\ \beta = r+t+rst,\ \gamma = rs/(r+t+rst)$$

so that

$$r = \beta\gamma/(\alpha+\gamma+\alpha\beta\gamma),\ s = \alpha+\gamma+\alpha\beta\gamma,\ t = \alpha\beta/(\alpha+\gamma+\alpha\beta\gamma).$$

Then, since j is odd, we can apply the previous case in reverse to get

$$\begin{aligned}
u_i(r)u_j(s)u_i(t) &= u_i(\beta\gamma/(\alpha+\gamma+\alpha\beta\gamma))u_j(\alpha+\gamma+\alpha\beta\gamma)u_i(\alpha\beta/(\alpha+\gamma+\alpha\beta\gamma))\\
&= u_j(\alpha)u_i(\beta)u_j(\gamma)\\
&= u_j(st/(r+t+rst))u_i(r+t+rst)u_j(rs/(r+t+rst)).
\end{aligned}$$

□

There are a few quick remarks we should make here. First, these relations are strikingly similar to the braid relations from Section 2. Also, these relations seem to suggest that EL_{2n} is somehow related to electrical networks. For example, relation (iii) is very reminiscent of the $Y-\Delta$ transformation. This relationship between EL_{2n} and electrical networks will be made precise in Theorem 5.3.

We now define the **nonnegative part** of EL_{2n}, denoted $(\mathrm{EL}_{2n})_{\geq 0}$, to be the subsemigroup of EL_{2n} generated by the set $\{u_i(t) : 1 \leq i \leq n,\, t \geq 0\}$.

Let $w \in S_{2n+1}$. For any reduced word $\underline{i} = i_1 \cdots i_k$ for w, we define a map $\psi_{\underline{i}} : \mathbb{R}^k_{>0} \to (\mathrm{EL}_{2n})_{\geq 0}$ by

$$\psi_{\underline{i}}(t_1,\ldots,t_k) = u_{i_1}(t_1)\cdots u_{i_k}(t_k).$$

Suppose that $\underline{j} = j_1 \cdots j_\ell$ is another reduced word for w. By Theorem 2.2, $\underline{i}$ and $\underline{j}$ are related by a sequence of braid moves. Then by applying the corresponding relations in Proposition 5.1, any element of the image of $\psi_{\underline{i}}$ is equal to an element of the image of $\psi_{\underline{j}}$, and vice-versa. This means that the images of $\psi_{\underline{i}}$ and $\psi_{\underline{j}}$ are equal. Thus, we define $E(w)$ to be the image of $\psi_{\underline{i}}$, which depends only on w. The sets $E(w)$ have the following properties.

Proposition 5.2. *$(\mathrm{EL}_{2n})_{\geq 0}$ is the disjoint union of the sets $E(w)$, and each of the maps $\psi_{\underline{i}} : \mathbb{R}^\ell_{>0} \to E(w)$ is a diffeomorphism.*

Let $\mathcal{P}(n+1)$ denote the set of response matrices of circular planar electrical networks with $n+1$ boundary vertices. We already know that if two circular planar electrical networks differ by a sequence of series, parallel, and $Y-\Delta$ transformations, then they have the same response matrix. Théorème 4 in [CdVGV96] states that if two circular planar electrical networks have the same response matrix, then they differ only by a sequence of series, parallel, and $Y - \Delta$ transformations. Thus, $\mathcal{P}(n+1)$ is equivalent to the set of circular planar electrical networks with $n+1$ boundary vertices modulo series, parallel, and $Y - \Delta$ transformations.

Theorem 5.3. $(\mathrm{EL}_{2n})_{\geq 0}$ *acts on* $\mathcal{P}(n+1)$.

Proof. Let Γ be a circular planar electrical network with $n+1$ boundary vertices, and label the boundary vertices $1, \ldots, n+1$. We will use the following notation

$$u_{2i-1}(t) \cdot \Gamma = \text{the electrical network obtained by attaching a boundary spike with conductivity } 1/t \text{ to } \Gamma \text{ at vertex } i,$$
$$u_{2i}(t) \cdot \Gamma = \text{the electrical network obtained by attaching a boundary edge with conductivity } t \text{ to } \Gamma \text{ between vertices } i \text{ and } i+1,$$

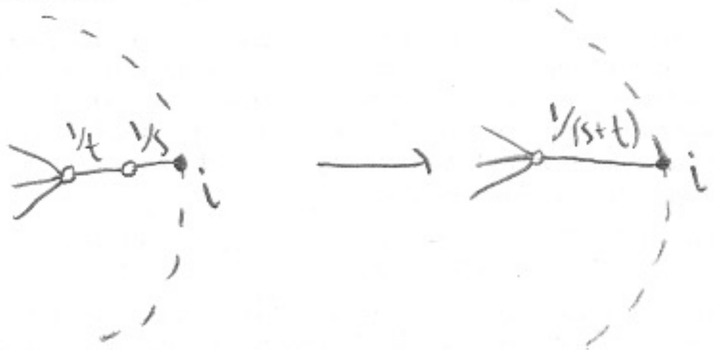
Notice that since conductivities must be positive real numbers, this notation doesn't quite make sense for $t = 0$. We would like to think of $u_{2i-1}(0) \cdot \Gamma$ as the result of attaching a spike with "infinite" conductivity (i.e., zero resistance) so that the endpoints of the spike may be identified, leaving us with a network that is essentially just Γ; similarly, $u_{2i}(0) \cdot \Gamma$ should correspond to attaching a boundary edge with no conductivity, leaving us again with a network that is essentially just Γ. With this in mind, we set $u_i(0) \cdot \Gamma = \Gamma$ for all i.

Now we can define

$$u_i(t) \cdot \Lambda(\Gamma) = \Lambda(u_i(t) \cdot \Gamma)$$

In order to extend this to give us an action of $(\mathrm{EL}_{2n})_{\geq 0}$ on $\mathcal{P}(n+1)$, we will first show that the $u_i(t) \cdot \Lambda(\Gamma)$ satisfy relations analogous to those in Proposition 5.1.

Notice that $u_{2i-1}(s) \cdot (u_{2i-1}(t) \cdot \Gamma)$ is the network obtained from Γ by attaching a boundary spike of conductivity $1/t$ at vertex i followed by another boundary spike of conductivity $1/s$ at vertex i. Applying a series transformation, we get the network obtained from Γ by attaching a single boundary spike of conductivity $1/(s+t)$ at vertex i.



Similarly, $u_{2i}(s) \cdot (u_{2i}(t) \cdot \Gamma)$ is the network obtained from Γ by attaching a boundary edge of conductivity t between vertices i and $i+1$ followed by another boundary edge of conductivity s between vertices i and $i+1$. Applying a parallel transformation, we get the network obtained from Γ by attaching a single boundary edge of conductivity $s+t$ between vertices i and $i+1$.

Therefore, by Proposition 1.1, for any i we have

$$\Lambda\big(u_i(s)\cdot(u_i(t)\cdot\Gamma)\big)=\Lambda\big(u_i(s+t)\cdot\Gamma\big).$$

Notice also that $u_i(s)\cdot(u_j(t)\cdot\Gamma)$ and $u_j(t)\cdot(u_i(s)\cdot\Gamma)$ are the same network whenever $|i-j|>1$. This is because, intuitively speaking, we are attaching boundary edges and/or spikes far enough apart from each other in the network that it doesn't matter which order we do them in. So,

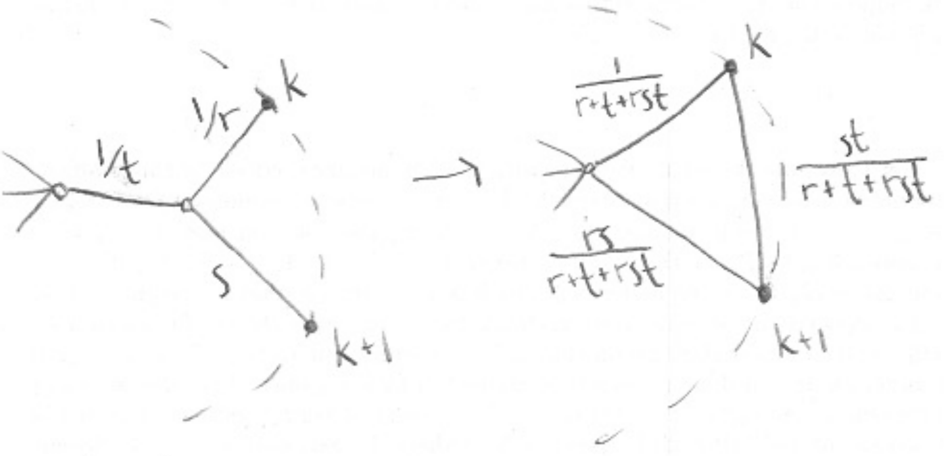
$$\Lambda\big(u_i(s)\cdot(u_j(t)\cdot\Gamma)\big)=\Lambda\big(u_j(t)\cdot(u_i(s)\cdot\Gamma)\big).$$

Now suppose $|i-j|=1$, and consider the networks

$$\Gamma'=u_i(r)\cdot(u_j(s)\cdot(u_i(t)\cdot\Gamma)),$$

$$\Gamma''=u_j(st/(r+t+rst))\cdot(u_i(r+t+rst)\cdot(u_j(rs/(r+t+rst))\cdot\Gamma)).$$

Observe that these networks are related by a $Y-\Delta$ transformation. For example, when $i=2k-1$ and $j=2k$, we have the following networks.



So, Γ′ and Γ″ are $Y-\Delta$ equivalent and hence

$$\Lambda(\Gamma')=\Lambda(\Gamma'').$$

Now consider $u_{i_1}(t_1)\cdots u_{i_k}(t_k)\in(\mathrm{EL}_{2n})_{\geq 0}$. Bby applying the relations of Proposition 5.1, we can rewrite this as $u_{j_1}(s_1)\cdots u_{j_\ell}(s_\ell)$, where $\underline{j}=j_1\cdots j_\ell$ is a reduced word. By applying the corresponding relations show above, we see that

$$\Lambda\big(u_{i_1}(t_1)\cdot\big(u_{i_2}(t_2)\cdot\big(\cdots\big(u_{i_k}(t_k)\cdot\Gamma\big)\cdots\big)\big)\big)=\Lambda\big(u_{j_1}(s_1)\cdot\big(u_{j_2}(s_2)\cdot\big(\cdots\big(u_{j_\ell}(s_\ell)\cdot\Gamma\big)\cdots\big)\big)\big).$$

Now let $u \in (\mathrm{EL}_{2n})_{\geq 0}$, and suppose we have two different decompositions for u: $u_{i_1}(t_1)\cdots u_{i_k}(t_k)$ and $u_{j_1}(s_1)\cdots u_{j_\ell}(s_\ell)$. Based on what we just showed in the preceding paragraph, let us assume that $\underline{i} = i_1 \cdots i_l$ and $\underline{j} = j_1 \cdots j_\ell$ are reduced words for w and v, respectively. Then $u \in \mathrm{im}(\psi_{\underline{i}}) \cap \mathrm{im}(\psi_{\underline{j}}) = E(w) \cap E(v)$, so by Proposition 5.2 we must have $w = v$. But this means that $\underline{i}$ and $\underline{j}$ are reduced words for the same permutation, so by Theorem 2.2 they must be related by a sequence of braid moves. This implies that $u_{i_1}(t_1)\cdots u_{i_k}(t_k)$ and $u_{j_1}(s_1)\cdots u_{j_\ell}(s_\ell)$ are related by a sequence of moves from Proposition 5.1, therefore

$$\Lambda\big(u_{i_1}(t_1)\cdot\big(u_{i_2}(t_2)\cdot\big(\cdots\big(u_{i_k}(t_k)\cdot\Gamma\big)\cdots\big)\big)\big)=\Lambda\big(u_{j_1}(s_1)\cdot\big(u_{j_2}(s_2)\cdot\big(\cdots\big(u_{j_\ell}(s_\ell)\cdot\Gamma\big)\cdots\big)\big)\big)$$

by applying the corresponding relations shown above.

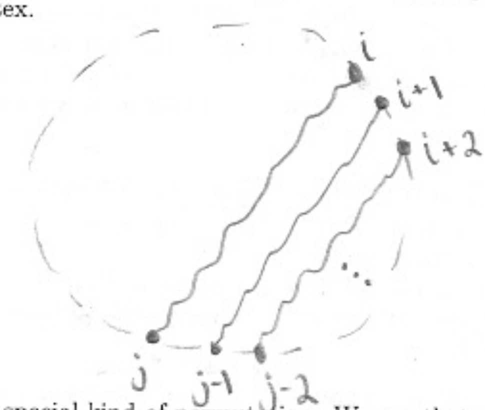
Now we define a map $(\mathrm{EL}_{2n})_{\geq 0} \times \mathcal{P}(n+1) \to \mathcal{P}(n+1)$ by

$$\begin{aligned} u\cdot\Lambda(\Gamma) &= u_{i_1}(t_1)\cdot\big(u_{i_2}(t_2)\cdot\big(\cdots\big(u_{i_k}(t_k)\cdot\Lambda(\Gamma)\big)\cdots\big)\big)\\ &= \Lambda\big(u_{i_1}(t_1)\cdot\big(u_{i_2}(t_2)\cdot\big(\cdots\big(u_{i_k}(t_k)\cdot\Gamma\big)\cdots\big)\big)\big), \end{aligned}$$

where $u = u_{i_1}(t_1)\cdots u_{i_k}(t_k)$. We have shown that this does not depend on our decomposition of u, therefore this map is well-defined. It is clear that this defines a group action of $(\mathrm{EL}_{2n})_{\geq 0}$ on $\mathcal{P}(n+1)$. □

Let Γ_0 denote the empty network with $n+1$ boundary vertices; by this we mean the network with no interior vertices and no edges. Given a permutation $w \in S_{2n+1}$, let $\underline{i} = i_1 \cdots i_k$ be a reduced word for w. Using the notation from the proof of Theorem 5.3, let Γ_w be the electrical network $u_{i_1}(1)\cdot(u_{i_2}(1)\cdot(\cdots(u_{i_k}(1)\cdot\Gamma)\cdots))$, and define G_w to be the underlying graph of Γ_w. Observe that although G_w does in fact depend on the reduced word chosen for w, any other choice of reduced word will result in a boundary graph that is $Y-\Delta$ equivalent to G_w: if $\underline{j} = j_1 \cdots j_k$ is another reduce word for w, then it is related to $\underline{i}$ by a sequence of braid relations by Theorem 2.2, so $u_{i_1}(1)\cdots u_{i_k}(1)$ and $u_{j_1}(t_1)\cdots u_{j_k}(t_k)$ are related by a sequence of relations (ii) and (iii) from Proposition 5.1 (where the parameters $t_1, \dots, t_k$ depend on these relations), and hence by our proof of Theorem 5.3, the underlying graph of $u_{j_1}(t_1)\cdots u_{j_k}(t_k)$ is $Y-\Delta$ equivalent to G_w.

Let G be a circular planar graph, and label the vertices $1, \dots, n+1$ in counterclockwise order. Given $1 \leq i < j \leq n+1$, we say that G is **$(\mathbf{i},\mathbf{j})$-connected** if there exists a sequence of disjoint paths $p_1, p_2, \dots, p_{\lfloor (j-i+1)/2 \rfloor}$ in G such that for each k, the path p_k starts at $i-1+k$ and ends at $j+1-k$ without passing through any other boundary vertex.



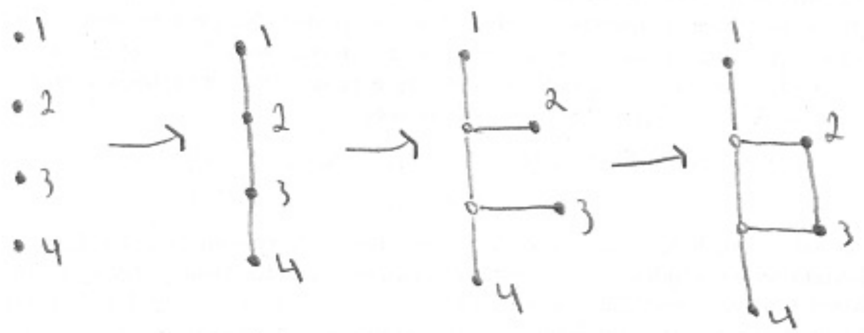
Now we introduce a special kind of permutation. We say that a permutation $w \in S_{2n+1}$ is **efficient** if

(i) $w(1) < w(3) < w(5) < \cdots < w(2n+1)$,
(ii) $w(2) < w(4) < w(6) < \cdots < w(2n)$,
(iii) $w(1) < w(2)$, $w(3) < w(4)$, $\ldots$, $w(2n-1) < w(2n)$.

Assertion. *Let $w \in S_{2n+1}$ be an efficient permutation. Then G_w is (i,j)-connected if and only if $w(2i) > w(2j-1)$.*

This claim is stated in [LP15] as a fact without proof. We will present a hand-wavy explanation as to why this should be true. For simplicity, let us take $i = 1$ and $j = 4$.

First suppose that our graph is $(1,4)$-connected. In order to create a path between 1 and 4, one must first create a path through the boundary by attaching boundary edges between 1 and 2, 2 and 3, and 3 and 4. In order to "interiorize" this path, one must attach boundary spikes at vertices 2 and 3. Lastly, in order to connect 2 and 3, we must attach a boundary edge between them.



This corresponds to the permutation $w = (\sigma_4)(\sigma_3\sigma_5)(\sigma_2\sigma_4\sigma_6)$. Observe that $w(2i) = w(2) = 5 > 4 = w(7) = w(2j-1)$.

Suppose instead that $w(2i) = w(2) > w(7) = w(2j-1)$. Then it must be that $\sigma_2\sigma_4\sigma_6$ is a factor of w. But this permutation inverts 3 and 4 and it inverts 5 and 6, so in order for w to remain an efficient permutation, we must multiply by $\sigma_3\sigma_5$. But this permutation inverts 4 and 5, so we must multiply by σ_4. This gives us the efficient permutation $(\sigma_4)(\sigma_3\sigma_5)(\sigma_2\sigma_4\sigma_6)$. As we described above, the permutation makes G_w $(1,4)$-connected.

The problem with this argument is that if G_w is $(1,4)$-connected, the the factors of the permutation $(\sigma_4)(\sigma_3\sigma_5)(\sigma_2\sigma_4\sigma_6)$ will be factors of w, but in general $w \neq (\sigma_4)(\sigma_3\sigma_5)(\sigma_2\sigma_4\sigma_6)$. One approach to prove this assertion is to try to use our knowledge of braid relations to try to prove the following: G_w is $(1,4)$-connected if and only if $i_1 \cdots i_k\, 4\,3\,5\,2\,4\,6$ is a reduced word for w for some $i_1, \dots, i_k$ if and only if $w(2i) > w(2j-1)$.

Proposition 5.4. *If the assertion above is true, then the following are true.*

(i) The map $\Theta_w : E(w) \to \mathcal{P}(n+1)$ define by $\Theta_w(u) = u \cdot \Lambda(\Gamma_0)$ is injective if and only if $w \in S_{2n+1}$ is efficient.
(ii) If w and v are distinct efficient permutations, then $\operatorname{im}(\Theta_w) \cap \operatorname{im}(\Theta_v) = \varnothing$.
(iii) If w is not efficient, then there is a unique efficient permutation v such that $\operatorname{im}(\Theta_w) = \operatorname{im}(\Theta_v)$.

Proof. (i) There is a partial ordering of S_{2n+1} given by $v \preceq w$ if and only if there exists $u \in S_{2n+1}$ such that $w = uv$ and $\operatorname{length}(w) = \operatorname{length}(u) + \operatorname{length}(v)$. It is a well-known fact that. It is a fact that $v \preceq w$ if and only if $w(i) > w(j)$ whenever $i < j$ and $v(i) > v(j)$, i.e., w inverts i and j whenever v does. Let $w^* \in S_{2n+1}$ be the efficient permutation

$$w^* = (s_{n+1})(s_n s_{n+2}) \cdots (s_4 s_6 \cdots s_{2n-2})(s_3 s_5 \cdots s_{2n-1})(s_2 s_4 \cdots s_{2n}).$$

Observe that w^* inverts every possible pair of indices that it can as an efficient permutation. Therefore, $w \preceq w^*$ for any efficient permutation w.

Now notice that G_{w^*} is the so-called standard graph of [CIM98], so by Proposition 7.3 and Theorem 2 of [CIM98], Θ_{w^*} is injective. Now let $w \in S_{2n+1}$ be any efficient permutation. Then since $w \preceq w^*$, there exists $u \in S_{2n+1}$ such that $w^* = uw$ and $\operatorname{length}(w^*) = \operatorname{length}(u) + \operatorname{length}(w)$. Let $i_1 \cdots i_k$ be a reduced word for u and $i_{k+1} \cdots i_\ell$ a reduced word for w. Then $w^* = uw = s_{i_1} \cdots s_{i_k} s_{i_{k+1}} \cdots s_{i_\ell}$ and $\ell = \operatorname{length}(w^*)$, so $i_1 \cdots i_k i_{k+1} \cdots i_\ell$ is a reduced word for w^*. Therefore, if Θ_w were not injective, then Θ_w^* would not be injective. Hence, Θ_w is injective.

Now suppose that w is not efficient. Then w must violate one of the three conditions given above for efficient permutations. If w violates either condition (ii) or (iii), then w inverts an odd index i with an index greater than i. By applying braid moves, there must exist a reduced word for w of the form $i_1 \cdots i_k i$. Then $u_i(t) \cdot \Gamma_0$ is Γ_0 with a single boundary spike attached, so we see that $\Lambda(u_i(t) \cdot \Gamma_0) = 0 = \Lambda(\Gamma_0)$. Therefore,

$$\begin{aligned} u_{i_1}(t_1) \cdots u_{i_k}(t_k) u_i(t) \cdot \Lambda(\Gamma_0) &= u_{i_1}(t_1) \cdots u_{i_k}(t_k) \cdot \Lambda(u_i(t) \cdot \Gamma_0) \\ &= u_{i_1}(t_1) \cdots u_{i_k}(t_k) \cdot \Lambda(\Gamma_0) \end{aligned}$$

tor all $t > 0$, hence Θ_w is not injective. If w violates condition (i), then w inverts an even index j with another even index greater than j. By applying braid moves, w must have a reduced word of the form $j_1 \cdots j_k (j+1) j$. Then by applying a series transformation, $u_{j+1}(s) u_j(t) \cdot \Gamma_0$ becomes $u_j(1/(s+t)) \cdot \Gamma_0$, so

$$\begin{aligned} u_{j_1}(t_1) \cdots u_{j_k}(t_k) u_{j+1}(s) u_j(t) \cdot \Lambda(\Gamma_0) &= u_{j_1}(t_1) \cdots u_{j_k}(t_k) \cdot \Lambda(u_{j+1}(s) u_j(t) \cdot \Gamma_0) \\ &= u_{j_1}(t_1) \cdots u_{j_k}(t_k) \cdot \Lambda(u_j(1/(s+t)) \cdot \Gamma_0) \\ &= u_{j_1}(t_1) \cdots u_{j_k}(t_k) \cdot \Lambda(u_j(1/(t+s)) \cdot \Gamma_0) \\ &= u_{j_1}(t_1) \cdots u_{j_k}(t_k) \cdot \Lambda(u_{j+1}(t) u_j(s) \cdot \Gamma_0) \\ &= u_{j_1}(t_1) \cdots u_{j_k}(t_k) u_{j+1}(t) u_j(s) \cdot \Lambda(\Gamma_0). \end{aligned}$$

Therefore, Θ_w is not injective.

(ii) Observe that since $w \neq v$, there must exist $1 \leq i, j \leq 2n+1$ such that $w(i) > w(j)$ but $v(i) < v(j)$. Without loss of generality, assume that $i < j$ (if $i > j$, switch w and v). Notice that since w is an efficient permutation,

$$\begin{aligned} i \text{ odd}, j \text{ odd} &\implies w(i) < \cdots < w(j) \\ i \text{ odd}, j \text{ even} &\implies w(i) < w(i+1) < \cdots < w(j) \\ i \text{ even}, j \text{ even} &\implies w(i) < \cdots < w(j). \end{aligned}$$

Therefore, i must be even and j must be odd. Write $i = 2k$ and $j = 2\ell - 1$ for some $1 \leq k < \ell \leq n+1$ so that $w(2k) > w(2\ell - 1)$ and $v(2k) < v(2\ell - 1)$. By the assertion, G_w is (k, ℓ)-connected and G_v is not, so G_w and G_v are not $Y - \Delta$ equivalent by Lemma 5.1 in [CIM98]. Now observe that for any $u \in E(w)$ and $u' \in E(v)$, the underlying graph of the network $u \cdot \Gamma_0$ is $Y - \Delta$ equivalent to G_w, and the underlying graph of the network $u' \cdot \Gamma_0$ is $Y - \Delta$ equivalent to G_v. Therefore, $u \cdot \Gamma_0$ and $u' \cdot \Gamma_0$ cannot be $Y - \Delta$ equivalent as networks, hence $u \cdot \Lambda(\Gamma_0) = \Lambda(u \cdot \Gamma_0) \neq \Lambda(u' \cdot G_0) = u' \cdot \Lambda(\Gamma_0)$. Thus, $\operatorname{im}(\Theta_w) \cap \operatorname{im}(\Theta_v) = \varnothing$.

(iii) If w is not efficient, then as explained in part (i), w has a reduced word $i_1 \cdots i_k$ such that either i_k is odd or i_k is even and $i_{k-1} = i_k + 1$. If i_k is odd, let $w' = s_{i_1} \cdots s_{i_{k-1}}$ so that, as we saw in part (i), $\operatorname{im}(\Theta_{w'}) = \operatorname{im}(\Theta_w)$. If i_{k-1} is even and $i_{k-1} = i_k + 1$, let $w' = i_1 \cdots i_{k-1} i_k$ so that $\operatorname{im}(\Theta_{w'}) = \operatorname{im}(\Theta_w)$. We can repeat this procedure for w' and so on. Once this process terminates, we will have found an efficient permutation v with $\operatorname{im}(\Theta_w) = \operatorname{im}(\Theta_v)$.

□

The following result again depends on the validity of the assertion above.

Corollary 5.5. $(\mathrm{EL}_{2n})_{\geq 0} \cdot \Lambda(\Gamma_0) = \{u \cdot \Lambda(\Gamma_0) : u \in (\mathrm{EL}_{2n})_{\geq 0}\}$ *is dense in* $\mathcal{P}(n+1)$.

6. Relation to the Temperley-Lieb Algebra.

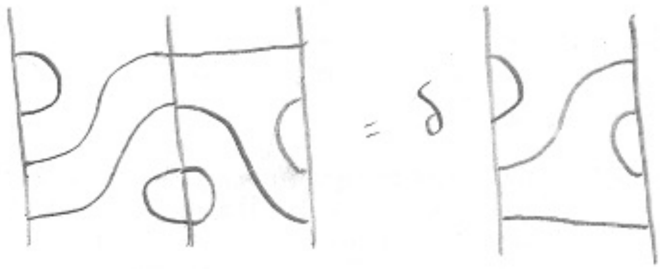
Let R be a commutative ring, and fix $\delta \in R$. The **Temperley-Lieb algebra** $TL_n(\delta)$ is the associative R-algebra generated by $U_1, \ldots, U_{n-1}$ subject to the relations

$$\begin{aligned} U_i^2 &= \delta U_i \quad \text{for all } \ i, \\ U_i U_j &= U_j U_i \quad \text{for } \ |i-j| > 1, \\ U_i U_j U_i &= U_i \quad \text{for } \ |i-j| = 1. \end{aligned}$$

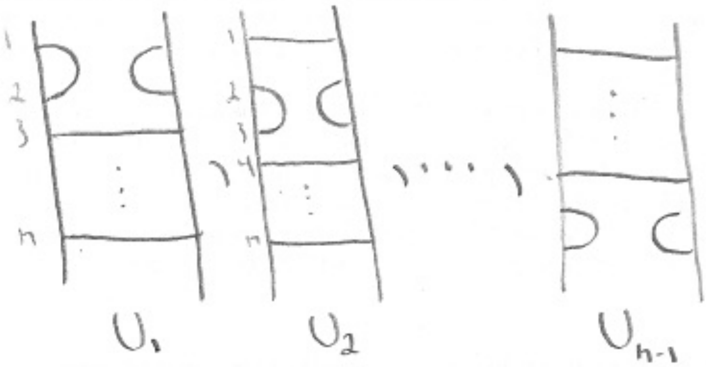
The Temperley-Lieb algebra is related to a wide range of subjects, including statistical mechanics, quantum field theory, and knot theory. A good introduction can be found in [Abr09].

The Temperley-Lieb algebra has a very nice visual representation in terms of planar diagrams. Suppose that we have two parallel rows of n points sitting in the plane, and suppose that we join the points pairwise with smooth arcs lying inside the rectangular region determined by the dots such that the arcs do not cross each other. Consider the set of formal R-linear combinations of all such diagrams (identified up to planar isotopy). Observe that this set has the structure of an associative R-algebra, where multiplication is defined in the following way:

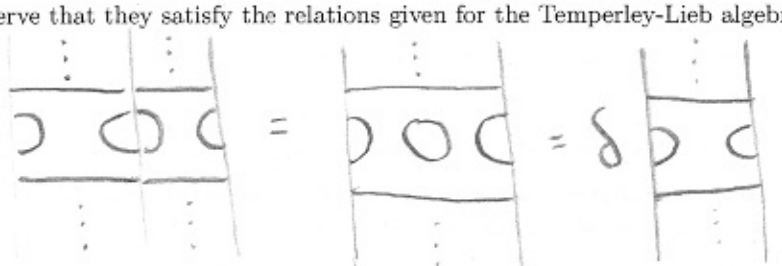
the product of two diagrams is the result of concatenating them and replacing each loop that is created with the scalar δ.

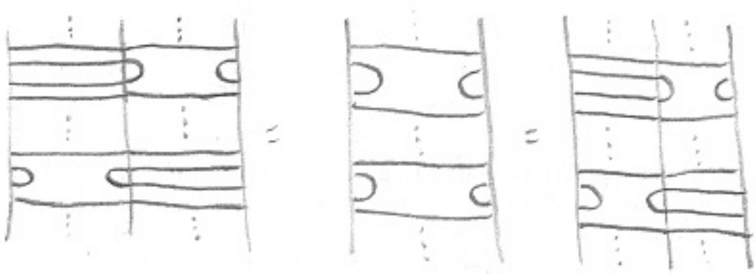


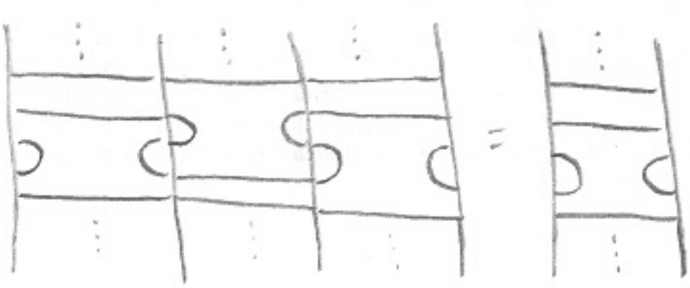
Now consider the following $n-1$ diagrams.



Observe that they satisfy the relations given for the Temperley-Lieb algebra.







Now we will describe a relationship between the Temperley-Lieb algebra and the electrical Lie algebra which was discovered by David Jekel. Take $R = \mathbb{R}$ and $\delta = 0$. Then $TL_{2n+1}(0)$ is the $\mathbb{R}$-algebra generated by $U_1, \ldots, U_{2n}$ subject to the relations

$$\begin{aligned} U_i^2 &= 0 \quad \text{for all} \quad i, \\ U_iU_j &= U_jU_i \quad \text{for} \quad |i-j| > 1, \\ U_iU_jU_i &= U_i \quad \text{for} \quad |i-j| = 1. \end{aligned}$$

We can give $TL_{2n+1}(0)$ a Lie algebra structure with the commutator bracket. Then for $|i-j| > 1$,

$$[U_i, U_j] = U_iU_j - U_jU_i = 0,$$

and for $|i-j| = 1$,

$$\begin{aligned} [U_i, [U_i, U_j]] &= U_i(U_iU_j - U_jU_i) - (U_iU_j - U_jU_i)U_i \\ &= U_i^2U_j - U_iU_jU_i - U_iU_jU_i + U_jU_i^2 \\ &= 0 - U_i - U_i + 0 \\ &= -2U_i. \end{aligned}$$

These are exactly the relations for $\mathfrak{el}_{2n}$.

References

[Abr09] S. Abramsky. Temperley-Lieb Algebra: From Knot Theory to Logic and Computation via Quantum Mechanics. *ArXiv e-prints*, October 2009.

[BB05] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

[CdVGV96] Yves Colin de Verdière, Isidoro Gitler, and Dirk Vertigan. Réseaux électriques planaires. II. *Comment. Math. Helv.*, 71(1):144–167, 1996.

[CIM98] E. B. Curtis, D. Ingerman, and J. A. Morrow. Circular planar graphs and resistor networks. *Linear Algebra Appl.*, 283(1-3):115–150, 1998.

[Gar02] A.M. Garsia. *The Saga of Reduced Factorizations of Elements of the Symmetric Group*. Publications du Laboratoire de Combinatoire et d'Informatique Mathématique. Université du Québec [Laboratoire de combinatoire et d'informatique mathématique (LACIM)], 2002.

[Hal15] Brian Hall. *Lie groups, Lie algebras, and representations*, volume 222 of *Graduate Texts in Mathematics*. Springer, Cham, second edition, 2015. An elementary introduction.

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[LP15] Thomas Lam and Pavlo Pylyavskyy. Electrical networks and Lie theory. *Algebra Number Theory*, 9(6):1401–1418, 2015.

[Su14] Y. Su. Electrical Lie Algebra of Classical Types. *ArXiv e-prints*, October 2014.

[Tit69] Jacques Tits. Le problème des mots dans les groupes de Coxeter. In *Symposia Mathematica (INDAM, Rome, 1967/68), Vol. 1*, pages 175–185. Academic Press, London, 1969.