# INFINITE GEODESICS AND CONNECTIVITY PROPERTIES IN PSEUDOCRITICAL INFINITE NETWORKS

GRACIE INGERMANSON

*Date*: 13 June 2013.

## 1. Definitions

**Definition 1.1.** A *circular planar graph* $G$ is a collection of boundary vertices $\partial V$, interior vertices $intV$ and undirected edges $E$ such that $|\partial V|, |intV|$ and $|E|$ are finite and $G$ has a circular embedding—that is, an embedding where every vertex in $\partial V$ is mapped to a point on the boundary circle, every vertex in $intV$ is mapped to a point in the open disk, and no pair of edges cross.

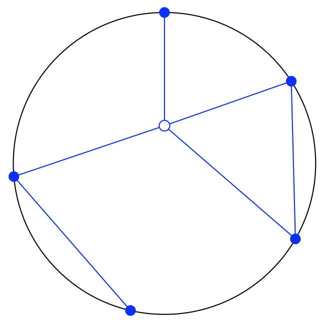


Figure 1.1—Example of a circular planar graph

**Definition 1.2.** A medial graph for a circular planar graph is a finite collection of smooth curves called geodesics, such that each geodesic has both its endpoints on the boundary circle.

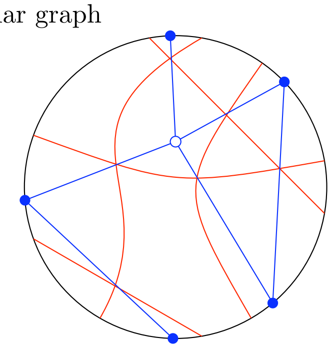


Figure 1.2—The medial graph for the given circular planar graph

Each medial graph admits exactly two 2-colorings—coloring each cell either black or white so that no cell in the medial graph shares an edge with another cell of the same color—each of which gives rise to a unique circular planar graph. The graph for a fixed 2-coloring is determined as follows: place a boundary vertex in each black cell adjacent to the boundary and an interior vertex in each black cell not adjacent to the boundary. Join vertices $v_1, v_2$ with an edge if and only if their respective medial cells share a corner.

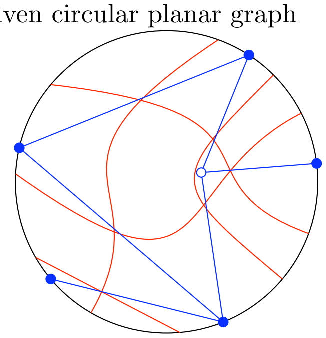


Figure 1.3—The *dual graph*—the circular planar graph corresponding to the reverse 2-coloring of the same medial graph.

**Definition 1.3.** An $infinite\, half{-}planar$ *network* $G$ is a collection of boundary vertices $\partial V$, interior vertices $intV$ and undirected edges $E$ that can be embedded in the upper half-plane such that the boundary vertices lie on the real axis, the interior vertices are in the open half-plane, and no pair of edges cross. We further require that every vertex $v$ be an endpoint of only finitely many edges.

**Definition 1.4.** A circular pair is a partition of the real line into two components $P$ and $Q$ such that some conformal mapping of the half-plane into the unit disk maps $P$ and $Q$ to arcs partitioning the boundary circle.

Figure 1.4—A circular pair formed by a partition of the real line into two rays

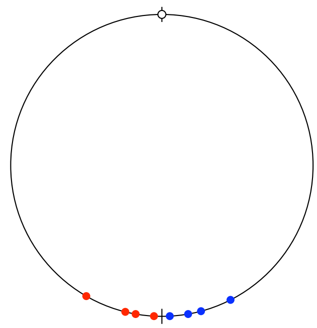


Figure 1.5—Under a conformal mapping to the unit disk, the point at infinity becomes an interior point in one of two arcs that partition the boundary circle.

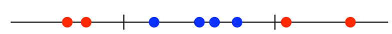


Figure 1.6—One component of the circular pair is a finite line segment, and the other is the union of two rays.

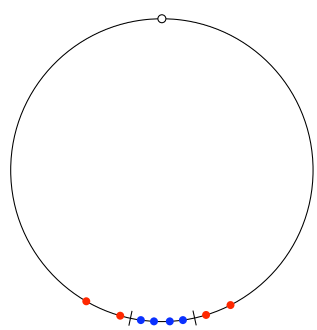


Figure 1.7—Under a conformal mapping to the unit disk, the point at infinity becomes an interior point in one of two arcs that partition the boundary circle.

**Definition 1.5.** A *connection* through an infinite half-planar network $G$ is a pair of finite sets of boundary vertices, $S_1$ and $S_2$, such that $|S_1| = |S_2|$ and $S_1$ and $S_2$ lie in complementary "arcs" with respect to some circular pair, together with a collection of vertex-disjoint paths where each path joins a vertex in $S_1$ to a vertex in $S_2$ and contains no other boundary vertices.

**Definition 1.6.** A graph $G$ is well-connected if given *any* finite sets of boundary vertices $S_1, S_2$ where $|S_1| = |S_2|$ and $S_1, S_2$ lie in complementary arcs of some circular pair, there exists a connection from $S_1$ to $S_2$.

**Definition 1.7.** To *delete* an edge $e = e_{v_1,v_2}$ from a network $G$ means to remove it from the edge set $E$ while leaving the rest of the edges in the graph unchanged.

**Definition 1.8.** To *contract* an edge $e = e_{v_1,v_2}$ means to replace $v_1$ and $v_2$ with a single vertex $\tilde{v}$ such that $\tilde{v}$ is adjacent to a vertex $p$ in the transformed graph if and only if $p$ was a vertex distinct from $v_1, v_2$ that was adjacent to at least one of $v_1, v_2$ in the original graph. We impose the constraint that edge contractions must preserve boundary vertices. That is, edges joining two boundary vertices may not be contracted. When contracting an edge joining a boundary vertex to an interior vertex, we identify the new vertex with the boundary vertex it replaces.

**Definition 1.9.** To *remove* an edge from a network $G$ means to either delete it or contract it.

**Definition 1.10.** We say that an edge removal (deletion or contraction) *breaks a connection* through $G$ if there are sets $S_1, S_2$ of boundary vertices that are connected through $G$ prior to the edge removal but not afterward.

3-connections with respect to each type of circular pair. (Edges in the graph omitted.)

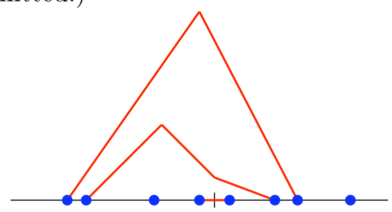


Figure 1.8

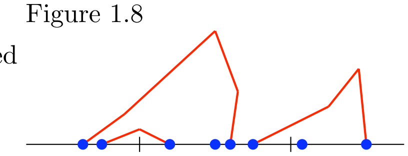


Figure 1.9

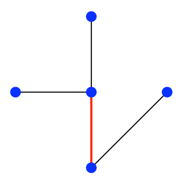


Figure 1.10—Original graph

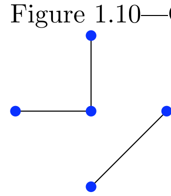


Figure 1.11—Graph after edge deleteion

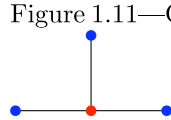


Figure 1.12—Graph after edge contraction

**Definition 1.11.** An edge $e$ is called *essential* if every legal method of removing it breaks a connection through $G$. That is, a boundary-boundary edge is essential if deleting it breaks a connection. If $e$ is not a boundary-boundary edge, it is essential if there exist connections $c_1, c_2$ such that $c_1$ is broken if $e$ is deleted and $c_2$ is broken if $e$ is contracted.

If an edge $e$ is not essential (that is, there is a legal way to remove it that preserves all connections through $G$), then it is called *inessential*.

**Definition 1.12.** A medial graph for an infinite half-planar network is a possibly infinite collection of smooth arcs (geodesics) in the upper half plane such that under a conformal mapping of the upper half-plane to the disk, each geodesic intersects the boundary circle twice. We further require that for any compact region in the upper half-plane, only finitely many geodesics pass through it.

**Definition 1.13.** A finite (or reentrant) geodesic is a geodesic that intersects the real axis twice.

**Definition 1.14.** A geodesic ray is an infinite geodesic that intersects the boundary exactly once.

**Definition 1.15.** A geodesic line is an infinite geodesic that does not intersect the boundary.

*Remark* 1.16. When we refer to a loop in a medial graph, we refer to a closed loop formed by a geodesic that intersects itself.

**Definition 1.17.** When a pair of geodesics intersect each other twice, we say they form a lens.

**Definition 1.18.** A pseudocritical network is an infinite network whose medial graph has no loops and no lenses.

*Remark* 1.19. Observe that two intersecting geodesic rays can be uncrossed to give either two non-intersecting geodesic rays or a finite geodesic and a geodesic line.

**Definition 1.20.** We say that a set of geodesics is parallel if no pair of geodesics in the set cross.

**Definition 1.21.** To comb a finite set of geodesic rays means to remove every intersection between pairs of rays in the set in the way that preserves geodesic rays.

*Remark* 1.22. Observe that combing a finite set of rays in a pseudocritical graph takes finitely many steps, because each uncrossing decreases the total number of crossings between pairs of geodesics in the set by one, and the initial number of crossings in a set of size k does not exceed $\binom{k}{2}$.

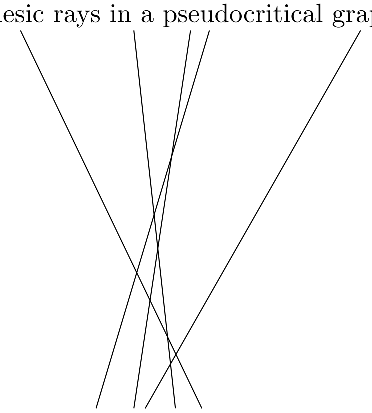
**Definition 1.23.** The geodesics in some finite set are called adjacent if the set consists of all the geodesics that have a least one endpoint in some fixed, finite, connected cut.

**Lemma 1.24.** *Let $S$ be a finite real interval such that if $g$ is any geodesic ray with an endpoint in $S$, $g$ does not bound any lens.*

*Then there is a finite sequence of uncrossings between pairs of rays with endpoints in $S$ such that* (1) *the number of rays with an endpoint in $S$ is the same at each step,* (2) *no step creates any lenses, and* (3) *after the process terminates, no pair of rays with an endpoint in $S$ cross.*

*Proof.* We first impose a partial ordering on the set $C$ of crossings between pairs of geodesic rays with endpoints in $S$. For each ray $g$ with an endpoint in $S$, fix a parametrization $\pi_g : [0,\ 1) \to \mathbb{H}$ such that $g$ is the image of $\pi_g$. We say that a crossing $c_1$ between a pair of rays with endpoints in $S$ is above another such crossing $c_2$ if there is a path from $c_1$ to $c_2$ using only segments of rays originating in $S$, where each segment is traversed in order with respect to its

Uncrossing a finite collection of geodesic rays in a pseudocritical graph:



Choose a "highest crossing": that is, identify two rays $g_1$ and $g_2$ such that $g_1$ and $g_2$ intersect at a point $c$, and if $f_1 : [0,1) \to g_1, f_2 : [0,1) \to g_2$ are parametrizations $f_1^{-1}(c') \leq f_1^{-1}(c)$ and parametrization.

Since there are finitely many rays with endpoints in $S$ and by hypothesis, no pair cross each other twice, there are finitely many crossings in $C$. Hence, there must be some crossing $c^*$ such that no crossing in $C$ is above $c^*$. We claim that uncrossing $c^*$ in the way that produces two rays does not introduce any lenses.

Suppose that $c^*$ is formed by the ray geodesics $g_1$ and $g_2$, and that uncrossing it replaces $g_1$ and $g_2$ with new ray geodesics $g_1'$ and $g_2'$. Since no other crossings are affected by the uncrossing, if any lens is produced, either $g_1'$ or $g_2'$ must bound it on one side. Assume that a lens is produced. Let $g_3$ denote the other geodesic bounding the lens. Since uncrossing $g_1$ and $g_2$ switches the portions above the crossing, $g_3$ must have crossed $g_1$ and $g_2$ on opposite sides of the crossing with respect to their parametrizations.

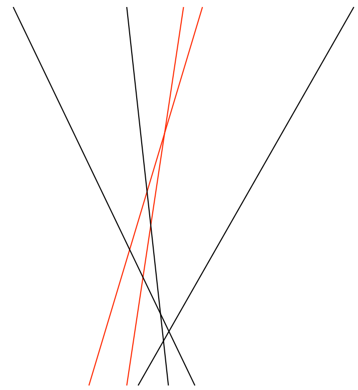
Without loss of generality, $g_3$ crossed $g_1$ below the crossing and $g_2$ above the crossing. By assumption, there were no crossings in $C$ above $c^*$, so the crossing between $g_2$ and $g_3$ cannot have been in $C$. Hence, $g_3$ is not a geodesic ray with an endpoint in $S$.

Consider the region $R$ bounded by $g_1$ and $g_2$ below $c^*$. Since $g_3$ does not have an endpoint in $S$, it enters and exits $R$ an equal number of times. Hence, it crosses $g_1 \cup g_2$ below $c^*$ an even number of times. By assumption, $g_3$ crosses $g_1$ below the crossing, so it crosses $g_1 \cup g_2$ below $c^*$ at least twice. But it also crosses $g_2$ above the crossing, so it crosses $g_1 \cup g_2$ at least 3 times. Hence, either $g_1$ or $g_2$ must cross $g_3$ multiple times, and hence form a lens with $g_3$. This is a contradiction, since no ray with an endpoint in $S$ bounds a lens.

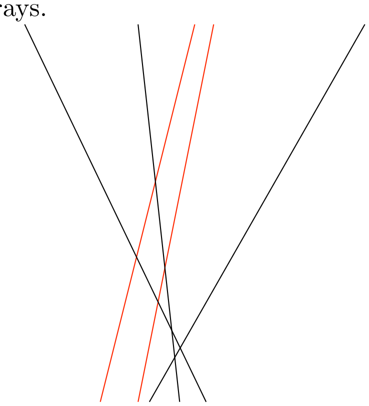
Hence, uncrossing a highest crossing does not introduce any lenses bounded by a geodesic with an endpoint in $S$.

Repeating this until there are no more crossings in $C$ shows that after combing the geodesic rays with endpoints in $S$, we

$f_2^{-1}(\tilde{c}) \leq f_2^{-1}(c)$ for any points $c'$ where $g_1$ crosses another geodesic from the collection and $\tilde{c}$ where $g_2$ crosses another geodesic from the collection.

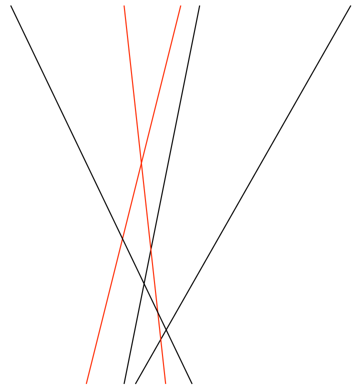


Remove the crossing as follows: remove the vertex $c$ from the medial graph. This splits $g_1$ into a finite half-geodesic and an infinite half-geodesic, and similarly for $g_2$. Attach the infinite portion of $g_1$ to the finite portion of $g_2$, and vice versa, separating the resulting geodesic rays.

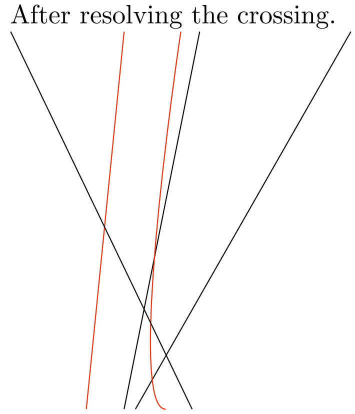


Choose a new highest crossing.

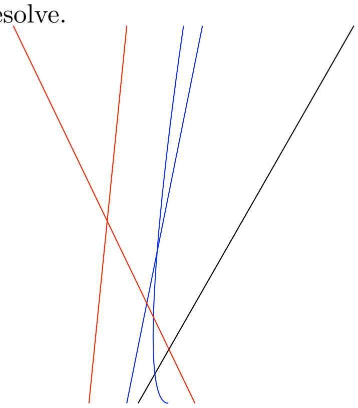
still have the property that no geodesic $g$ with an endpoint in $S$ bounds any lens. ☐



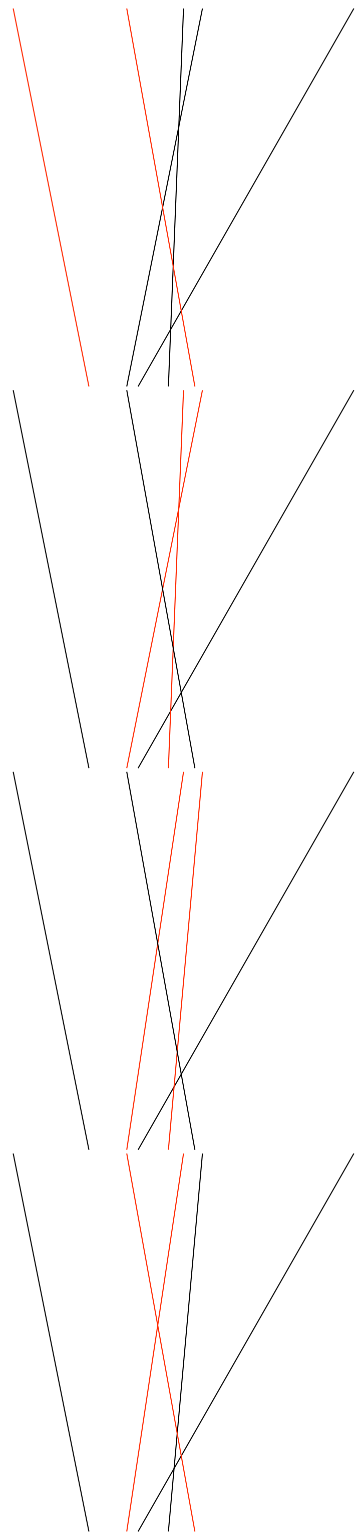
After resolving the crossing.

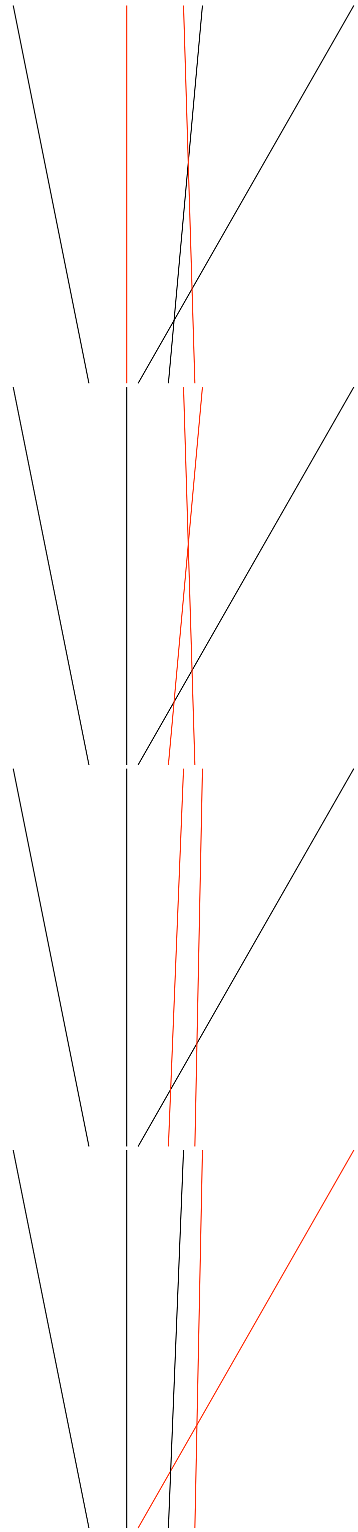


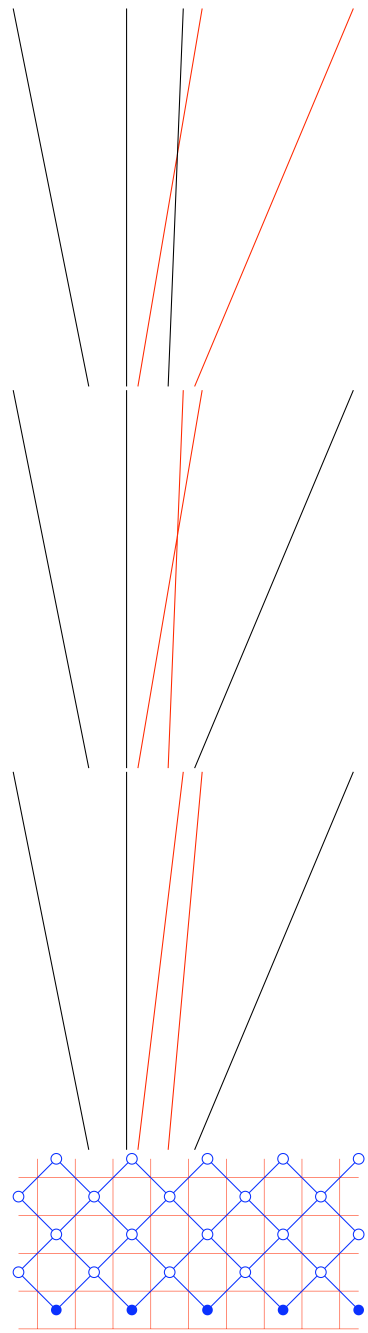
Note that it is possible for a collection of rays to have two "highest crossings" (shown in red and blue). In such a case, choose an arbitrary highest crossing to resolve.



Crossing resolved.







**Definition 1.25.** A $half{-}planar\, lattice\, graph$ is an infinite network whose medial graph can be drawn as a lattice: that is, the medial graph consists of infinitely many ray geodesics and infinitely many line geodesics, such that no pair of rays cross, no pair of lines cross, and each line crosses each ray exactly once.

**Definition 1.26.** A $lattice\, block$ is a finite, circular planar graph that is equivalent to the graph formed as follows. Take $2k$ consecutive rays and the lowest $k$ lines intersecting them. Take the subgraph bounded by the real axis, the left- and rightmost selected rays, and the $k^{th}$ horizontal line, and remove all edges exiting the region.

*Remark* 1.27. Consider the lattice graph with medial graph consisting of infinitely many vertical rays intersected by infinitely many horizontal lines. Consider any set of $k$ adjacent boundary vertices (where $k$ is finite). Let S be the finite subgraph bounded between the vertical geodesic left of the leftmost boundary vertex, the vertical geodesic right of the rightmost boundary vertex, and the $k^{th}$ horizontal geodesic from the boundary and preserving boundary and interior nodes. Clearly, S is well-connected.

*Proof.* Suppose that the boundary vertices in the lattice block are labeled from left to right as $v_1$, $v_2$, ..., $v_k$. Suppose that we wish to connect the vertices $v_a$ and $v_{a+d}$, where $a, d > 0$ and $a + d \leq k$. Travel up and to the right $d$ times, then down and to the right $d$ times. □

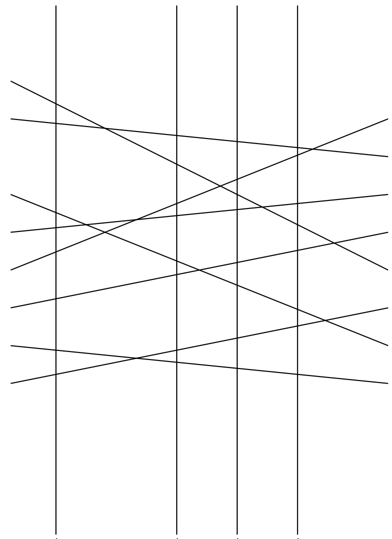
*Remark* 1.28. Since removing edges does not introduce any new connections, if a network $G$ can be transformed to a network $\tilde{G}$ via a series of edge-removals, then every connection in $\tilde{G}$ is also present in $G$.

**Lemma 1.29.** *Let $G$ be a pseudocritical half-planar graph and let $S \subseteq G$ be a connected real interval with $k$ boundary vertices. Suppose that the $2k$ geodesic endpoints about these vertices belong to distinct, parallel geodesic rays. If there are at least $k$ disjoint, broken-line geodesic paths across these rays such that each path intersects each ray at a single point, then $S$ is well-connected.*

*Proof.* We will show that the broken-line paths can be uncrossed to give at least $k$ straight geodesic paths across the parallel, adjacent rays. Because this gives a

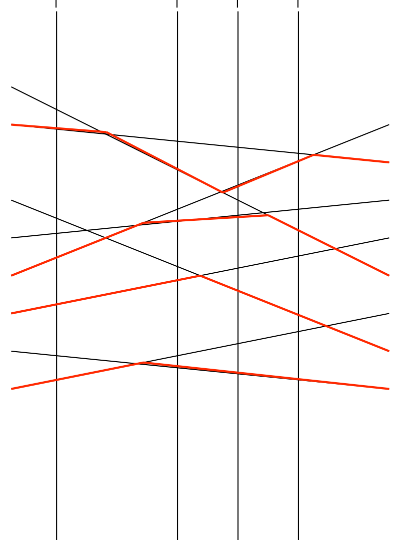
Let $I$ be a finite real interval. Suppose there are $n$ geodesic endpoints in $I$, belonging to $n$ geodesic rays such that no pair cross each other and none of these rays bounds a lens. Let $l$ denote the leftmost of these rays, and let $r$ denote the rightmost. If there are $n$ disjoint, broken-line geodesic paths across the region between $l$ and $r$, ...

lattice block, which we know to be well-connected, it will imply that the boundary nodes in the original strip was well-connected. Since we restrict our definition of infinite networks to those graphs such that only finitely many geodesics intersect any compact subset of the medial graph, the paths can be uncrossed in finitely many steps. Furthermore, this will not introduce any crossings of the rays with endpoints in the interval, since by assumption the broken-line paths do not use any segments of the parallel rays.



Let $R$ be the compact region bounded by the leftmost and rightmost geodesics in the cut, the real axis, and the $k^{th}$ broken-line path counted from the bottom.

Let $p$ be the lowest path in the set. By the Jordan curve theorem, since $R$ is a simply connected planar region, $p$ separates $R$ into two disjoint regions.



Follow the path across the upper border of the lowest layer of cells in $R$ and remove any crossings along it in the way that preserves the path. This process produces a horizontal path that intersects each geodesic ray from the cut at its lowest crossing. Clearly, the lowest layer of geodesic cells in $R$ is below $p$ and paths 2 through $k$ are above $p$. Since the $k-1$ disjoint paths above $p$ are not affected by the process, we can repeat this inductively, treating the last path uncrossed taking the role of $\mathbb{R}$.

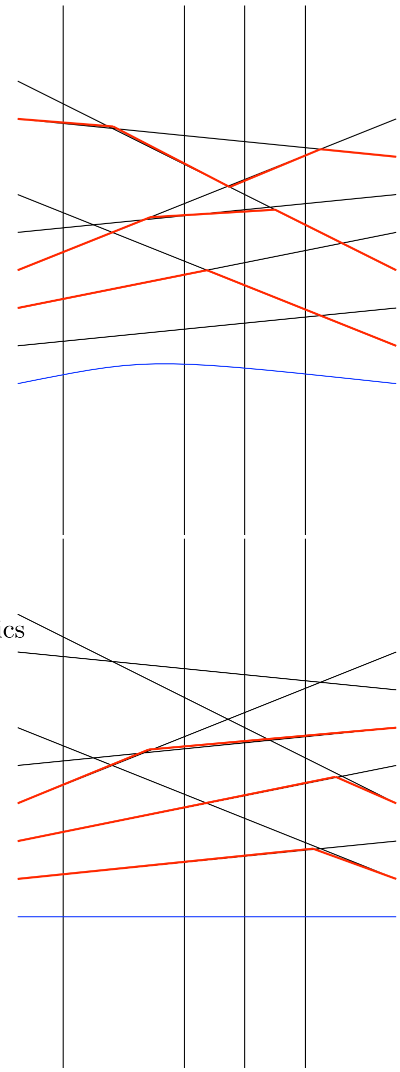
After $k$ iterations, we are left with $k$ disjoint, horizontal paths across the region such that for each $1 \leq i \leq k$, the $i^{th}$ path intersects each geodesic ray from the cut at its $i^{th}$ crossing.

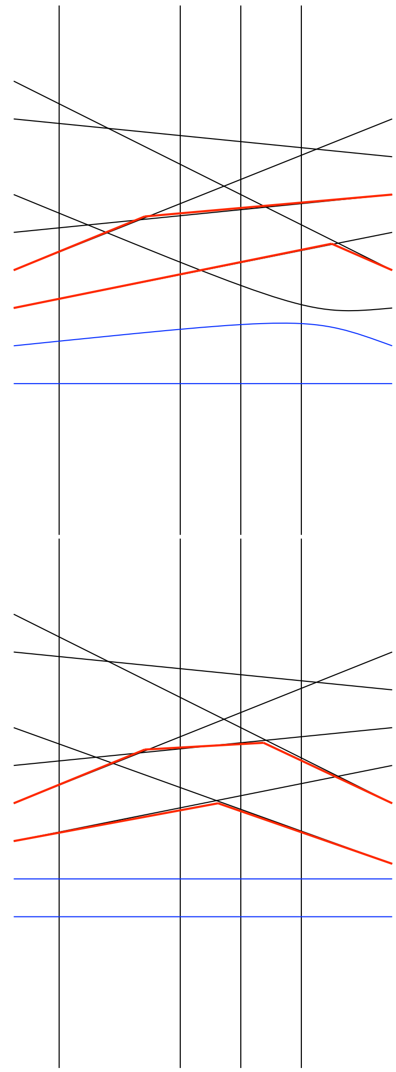
These paths form a lattice block with the parallel rays from the cut. $\square$

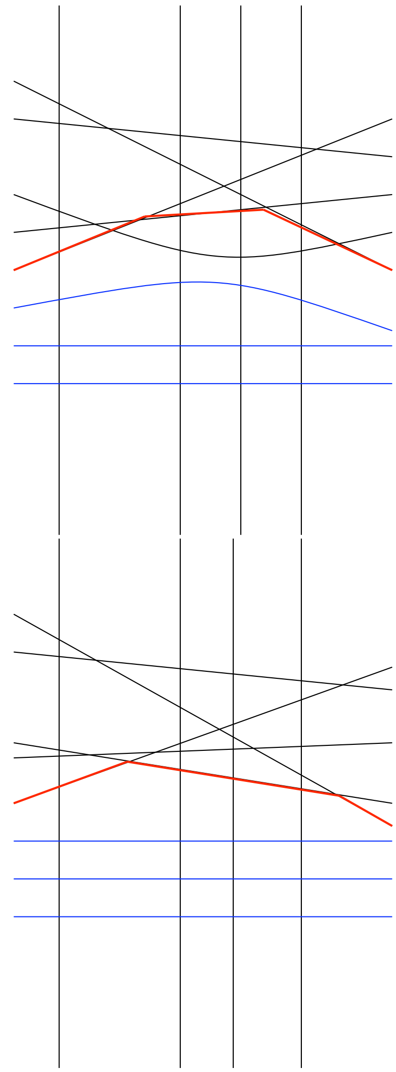
## 2. Theorems and results

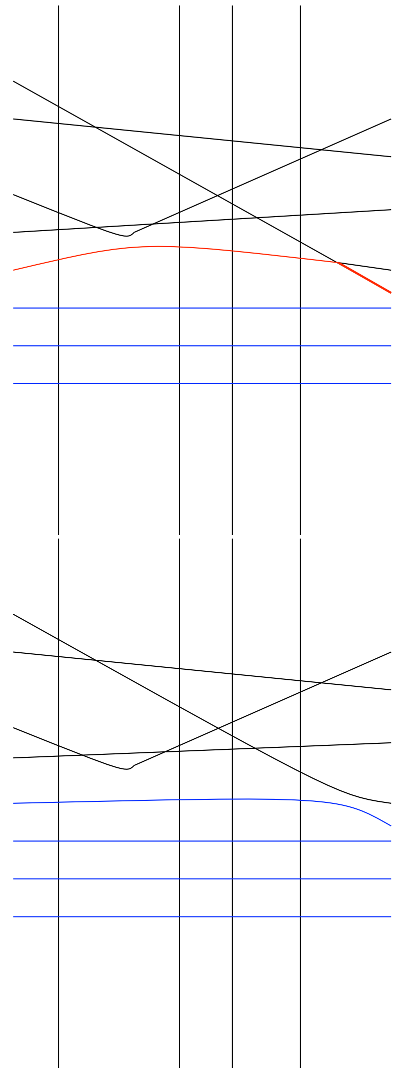
**Theorem 2.1.** *Every pseudocritical graph with compact medial cells and no reentrant geodesics is well-connected.*

*Proof.* Given any two finite sets of boundary vertices contained in disjoint arcs, there is a finite, connected interval of $\mathbb{R}$ containing their union. It therefore suffices to show that the set of boundary vertices along any finite, connected strip is well-connected. Choose any such set. The geodesic endpoints in the cut belong to a finite set of adjacent geodesic rays. Combing these rays makes them a set of adjacent, parallel rays through a sequence of edge removals (which cannot create new connections). By compactness of medial cells, there must have been infinitely many disjoint, broken-line geodesic paths $p$ such that for each geodesic ray $g$ originating in the cut, $p$ crossed $g$ above the last crossing of $g$ with some geodesic ray from the cut. Since no intersections of cut-rays and other geodesics are removed in the combing, and the combed graph is lensless, it must have infinitely many disjoint, broken-line paths across the parallel rays. Hence, the strip has at least $k$ disjoint, broken-line geodesic paths across its parallel rays, and so it is connected. □









**Corollary 2.2.** *Every edge crossing two ray geodesics in a pseudocritical graph with compact medial cells and no reentrant geodesics is inessential.*

*Proof.* Choose any edge crossing two boundary rays, and comb the (inclusive) set of rays between them. The resulting graph lacks the edge in question and is pseudocritical with compact medial cells and no reentrant geodesics. Hence, it is well-connected, and so no connection was broken. □

**Definition 2.3.** A half-geodesic is one of the connected components obtained by removing a vertex from the medial graph.

**Theorem 2.4.** *Every edge in a pseudocritical graph with compact medial cells and no reentrant geodesics is inessential.*

*Proof.* It suffices to show that any edge crossing at least one line geodesic is inessential. We claim that after removing the edge, given any finite set of adjacent boundary vertices, there is a sequence of geodesic uncrossings resulting in a graph in which that set is clearly well-connected.

Choose any pair of crossing geodesics in which at least one is a line geodesic and any finite cut with $k$ adjacent boundary vertices and their corresponding geodesic rays. Let $U$ be the union of the set of geodesic rays in the cut and the chosen crossed geodesics. Introduce a fictitious reentrant geodesic $g$ such that the region below $g$ contains all crossings between pairs of geodesics in $U$. This cell contains finitely many crossings. Introduce another fictitious reentrant geodesic $\tilde{g}$ such that the region below $\tilde{g}$ contains if $u$ and $\tilde{u}$ are geodesics in $U$ and $v$ is a geodesic that intersects $u$ in the region bounded by $g$, then either $\tilde{u}$ and $v$ intersect under $\tilde{g}$ or they do not intersect.

Observe that by compactness of medial cells, there are infinitely many disjoint, broken-line geodesic paths that cross every half-geodesic from $U$ that exits the region bounded by $\tilde{g}$.

Remove every crossing between pairs of geodesics in $U$ in a way that preserves infiniteness. (This can be completed in no more than $\binom{k+2}{2}$ steps.) This process may produce infinitely many lenses, but observe that every lens bounded on one side by one of the geodesic rays from the cut occurs in the region below $\tilde{g}$. This is because uncrossing a pair of geodesics introduces a lens only if there was some geodesic which crossed both on opposite sides of the crossing.

Since the lenses are compact, each can be emptied in finitely many steps, then removed by uncrossing both poles of every such lens in the proper way (such that if $g_1, g_2$ are geodesics forming a lens, the portion of $g_1$ below the first crossing in the lens is connected to the portion of $g_1$ above the second crossing). At this point, we can make the same argument about uncrossing paths to show equivalence to a lattice block. □

*Remark* 2.5. Suppose $g$ is a finite geodesic. Then $g$ partitions $\mathbb{H}$ into a bounded region $B(g)$ and an unbounded region $U(g)$.

**Definition 2.6.** A network $G$ is called locally bounded if $\forall v \in G, \exists$ a finite geodesic $g$ such that $v \in B(g)$.

**Lemma 2.7.** *A pseudocritical network with compact medial cells is locally bounded if and only if its dual graph is locally bounded.*

*Proof.* It suffices to prove one direction, since duality is a symmetric property. Suppose that $G$ is a pseudocritical, locally bounded network with compact medial cells, and choose any dual cell $d$. Assume that $d$ is not bounded by any finite geodesic. Since the primal cells bordering $d$ are bounded by finite geodesics, $d$ must be surrounded by some collection $C$ of finite geodesics $g$ such that $d \subseteq U(g)$ for each $g$ (and possibly by some finite segment of $R$). Let $g_1$ be the geodesic in $C$ with the leftmost endpoint. Consider the first geodesic $g_2$ in $C$ that intersects $g_1$ under some parametrization such that the left endpoint of $g_1$ is its initial point. Let $l_1, r_1$ be the left and right endpoints of $g_1$, respectively, and let $l_2, r_2$ be the left and right endpoints of $g_2$. Now $r_2$ must be to the right of $r_1$, since otherwise both $l_2$ and $r_2$ would be between $l_1$ and $r_1$. Hence the parity of the number of crossings of $g_1$ and $g_2$ is even. Since we know they cross, they must form a lens, a contradiction. Similarly, if $l_2$ is to the right of $r_2$, $g_1$ and $g_2$ form a lens. Hence, the endpoints must have the order $l_1, l_2, r_1, r_2$. $d$ is compact, bordered only by geodesics in $C$, and the intersection with $g_2$ is the first crossing between $g_1$ and another geodesic in $C$, so $d$ must be bordered by some segment of $g_1$ occurring after the crossing. But this implies that $d \in B(g_2)$, contradicting the unboundedness of $d$.

Hence, in a pseudocritical network with compact medial cells, if every primal cell is bounded by a finite geodesic, then every dual cell is bounded. □

**Lemma 2.8.** *Let $G$ be a pseudocritical, connected network with compact medial cells. The following are equivalent: (i) $G$ is locally bounded; (ii) there exists some vertex $v \in G$ such that $v \in B(g)$ for infinitely many finite geodesics $g$; (iii) every vertex $v \in G$ is in $B(g)$ for infinitely many finite geodesics $g$.*

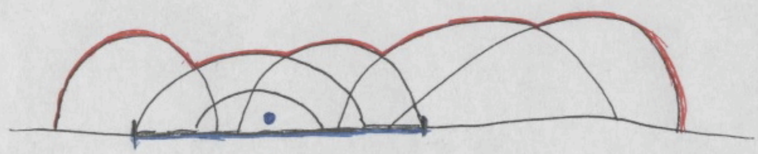
*Proof.* Clearly, (iii) implies (i) and (ii).

$(i) \implies (ii)$: Suppose that $G$ is locally bounded. Choose any vertex $v \in G$, and suppose that $v$ is bounded by exactly $k$ finite geodesics for some $k < \infty$. Let $S$ be the smallest connected subset of $\mathbb{R}$ containing all the endpoints of these $k$ bounding geodesics. We claim that there exists some finite geodesic $h$ such that one endpoint of $h$ is to the left of $S$ and the the other endpoint is to the right of $S$. Since $G$ is pseudocritical, this implies that $h$ bounds every finite geodesic with both endpoints in $S$.

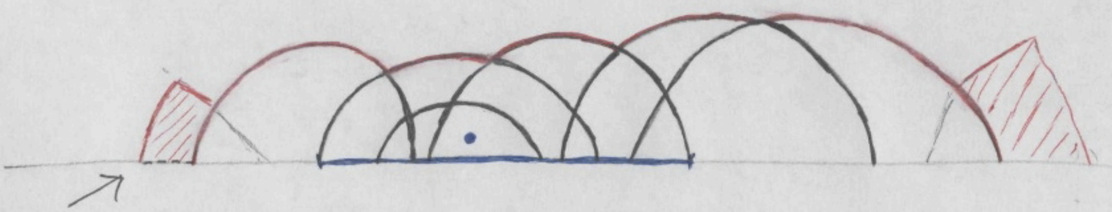
Let $F$ be the compact region bounded by the union of all finite geodesics with at least one endpoint in $S$. By compactness of medial cells, there must be a nonempty layer $L$ of cells such that $\forall c \in L, c \cap F \neq \emptyset$ and $c \cap F \subseteq \partial F$. Since G is locally bounded, each such $c$ must be bounded by some finite geodesic $g_c$.

Order the cells in $L$ starting from the cell bordering $\mathbb{R}$ and containing the leftmost geodesic endpoint in $F$. Observe the following: the first cell in the ordering can only be bounded by a geodesic with an endpoint to the left of $S$, and the last cell can only be bounded by a geodesic with an endpoint to the right of $S$. It is impossible to have two consecutive cells such that one is bounded by a geodesic with both endpoints to the left of $S$ and the other by a geodesic with both endpoints to the right of $S$, since this would imply a lens. Hence, at least one cell in $L$ must be bounded by a geodesic with one endpoint to the left of $S$ and one to the right of $S$, proving our claim.

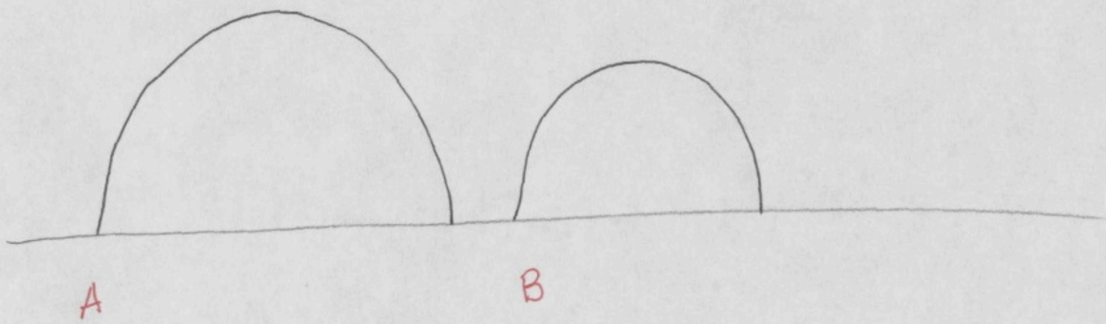
$(ii) \implies (iii)$: Suppose that some vertex $v \in G$ is bounded by infinitely many finite geodesics. Choose any vertex $\tilde{v} \in G$. Then there exists some compact region $C$ containing both $v$ and $\tilde{v}$. Since only finitely many of the geodesics bounding $v$ can pass through $C$ and $v$ is bounded by infinitely many geodesics, there must be infinitely many geodesics bounding $C$. Hence, $\tilde{v} \in C$ is bounded by infinitely many finite geodesics. □

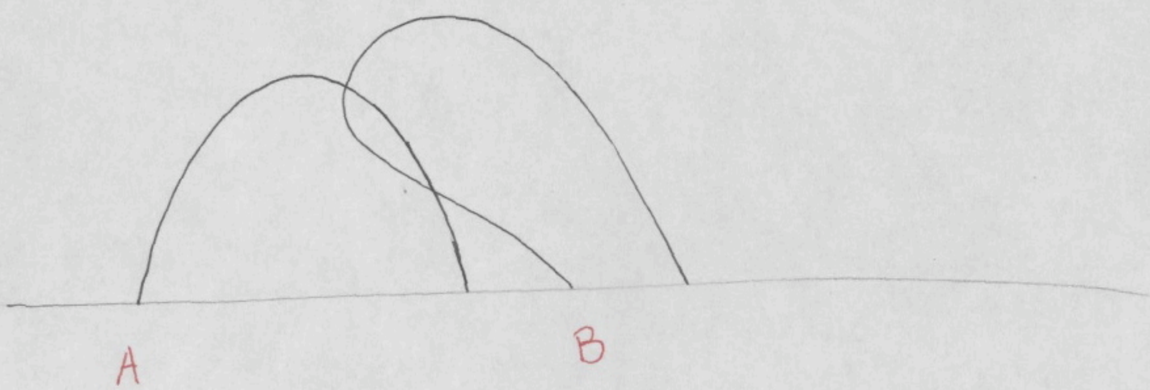


Suppose that the blue interval contains all endpoints of all finite geodesics bounding the blue vertex, and the red path is the boundary of the region bounded by the finite geodesics with at least one endpoint in the blue interval.

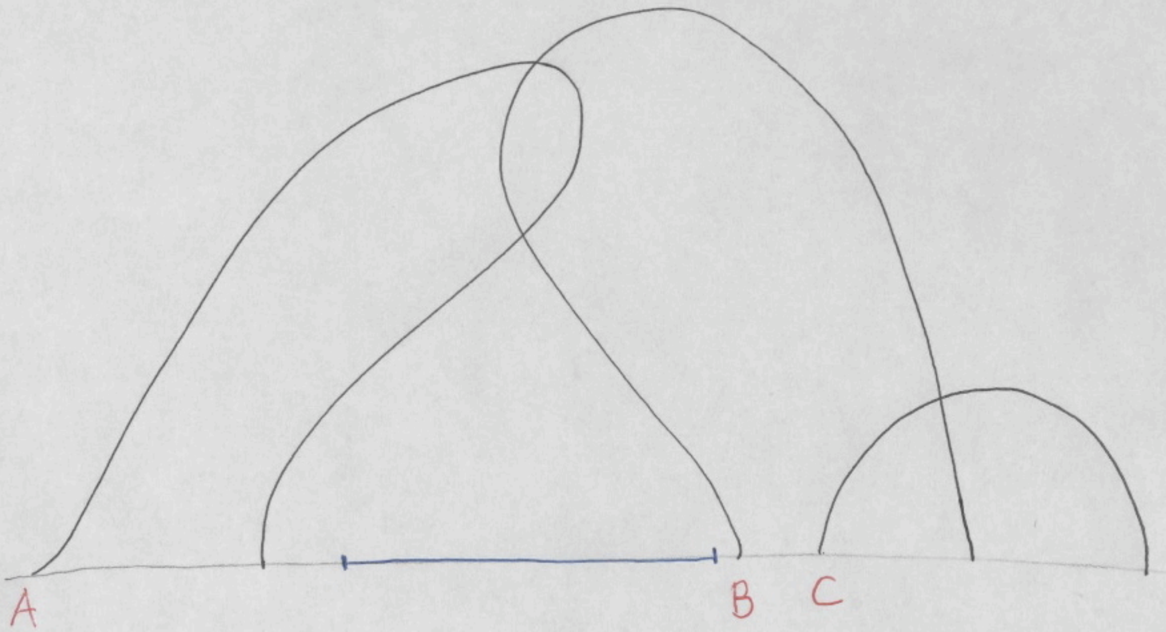


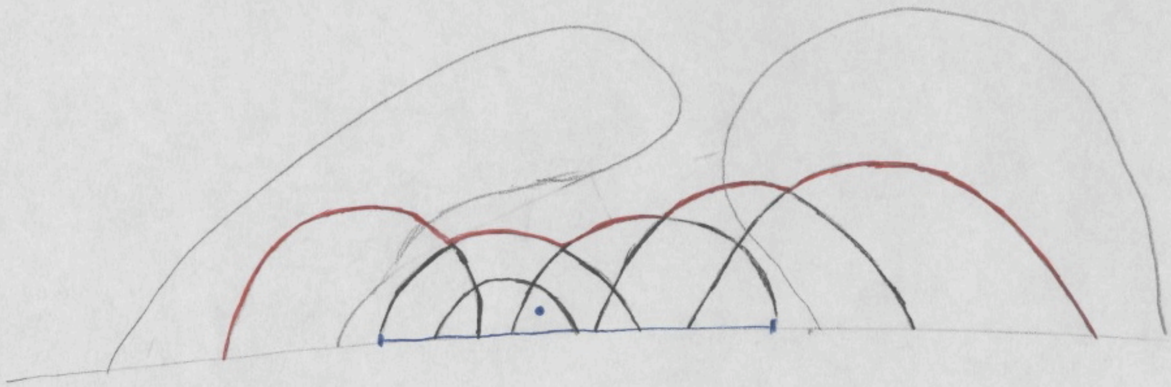
Any finite geodesic bounding this cell bounds its intersection with $\mathbb{R}$.
Hence, its left endpoint must be to the left of the blue interval.



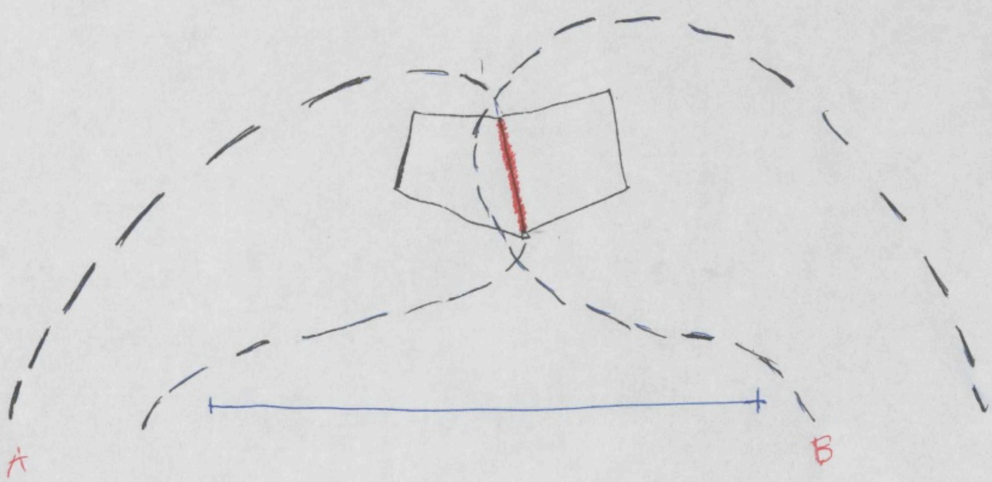

If A and B are finite geodesics such that both endpoints of A are to the left of both endpoints of B, then A and B intersect an even number of times,


A
B
C



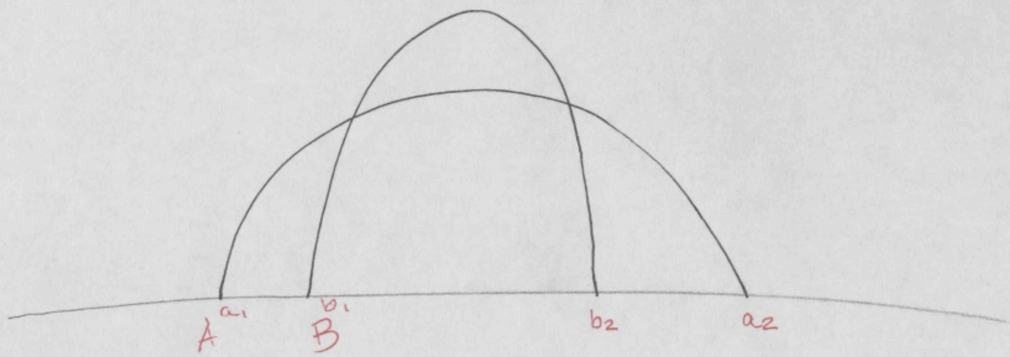
At least one cell adjacent to the red path but not bounded by it cannot be bounded by any finite geodesic with both endpoints on the same side of the blue interval.

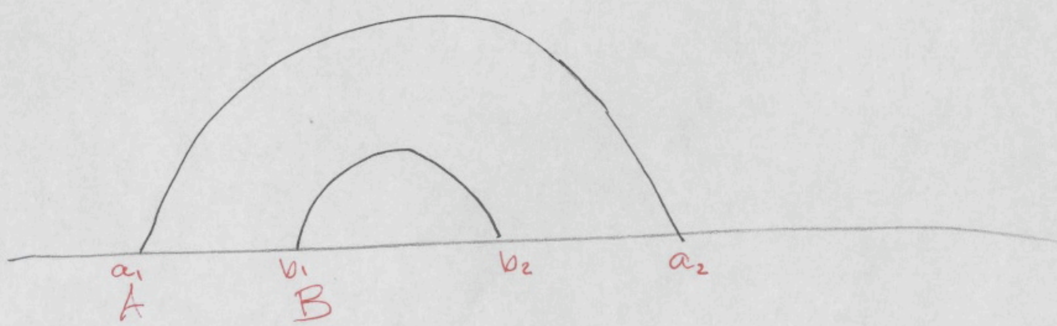
By local boundedness, there must be a finite geodesic whose endpoints are on opposite sides of the blue interval,




Two cells that share an edge cannot be bounded by finite geodesics A and B such that the right endpoint of A is to the left of the blue interval and the left endpoint of B is to the right of the blue interval.

If the shared edge is in $B(A) \cap B(B)$, then $B(A) \cap B(B) \neq \emptyset$ and hence A and B form a lens.




If A and B are finite geodesics and both endpoints of B are between the endpoints of A, then A and B intersect an even number of times.

**Theorem 2.9.** *Let $G$ be a pseudocritical network with compact medial cells. Every edge crossing two infinite geodesics is inessential.*

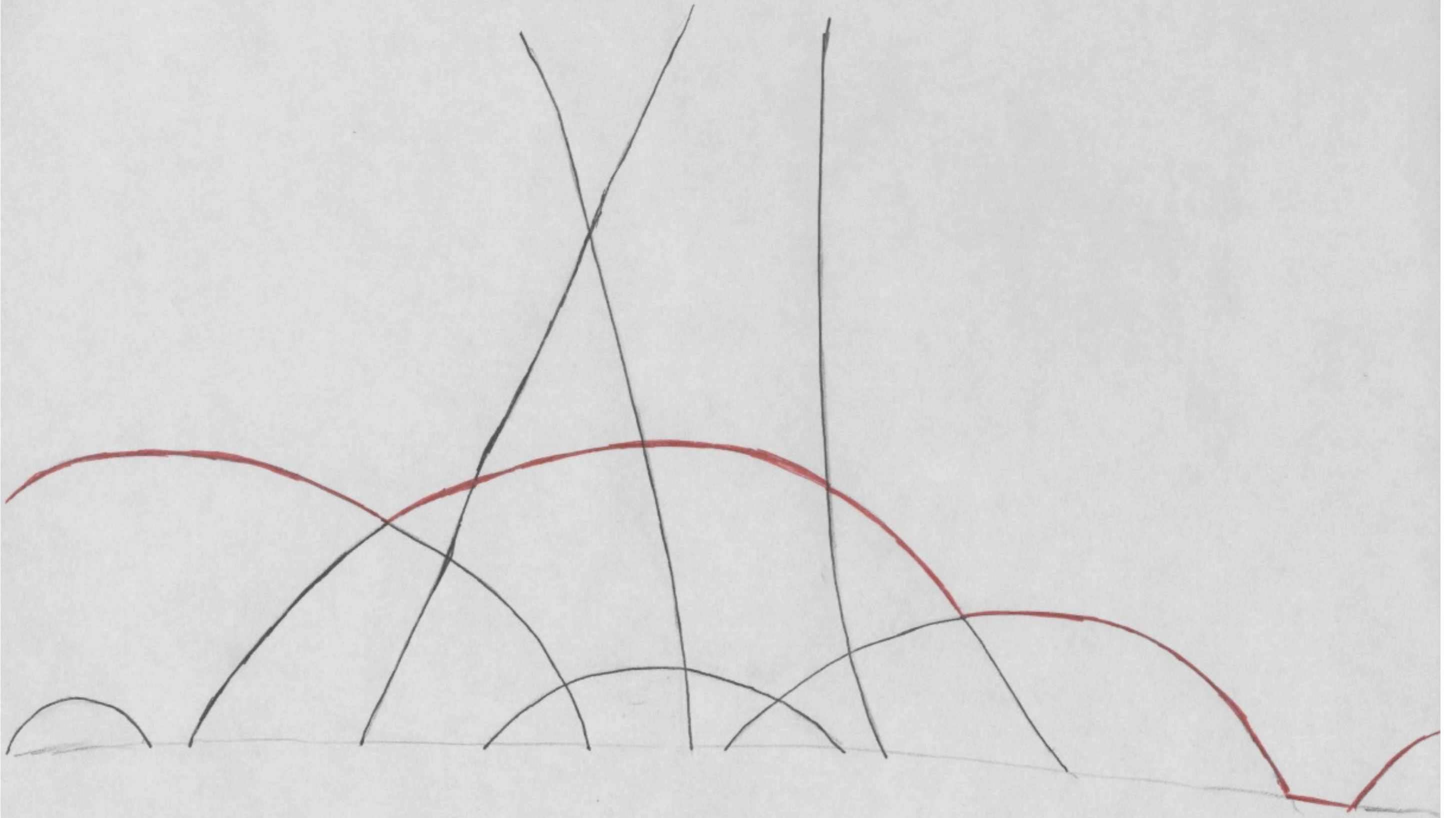
*Proof.* Let $c$ be a crossing of two infinite geodesics. There are 3 cases.

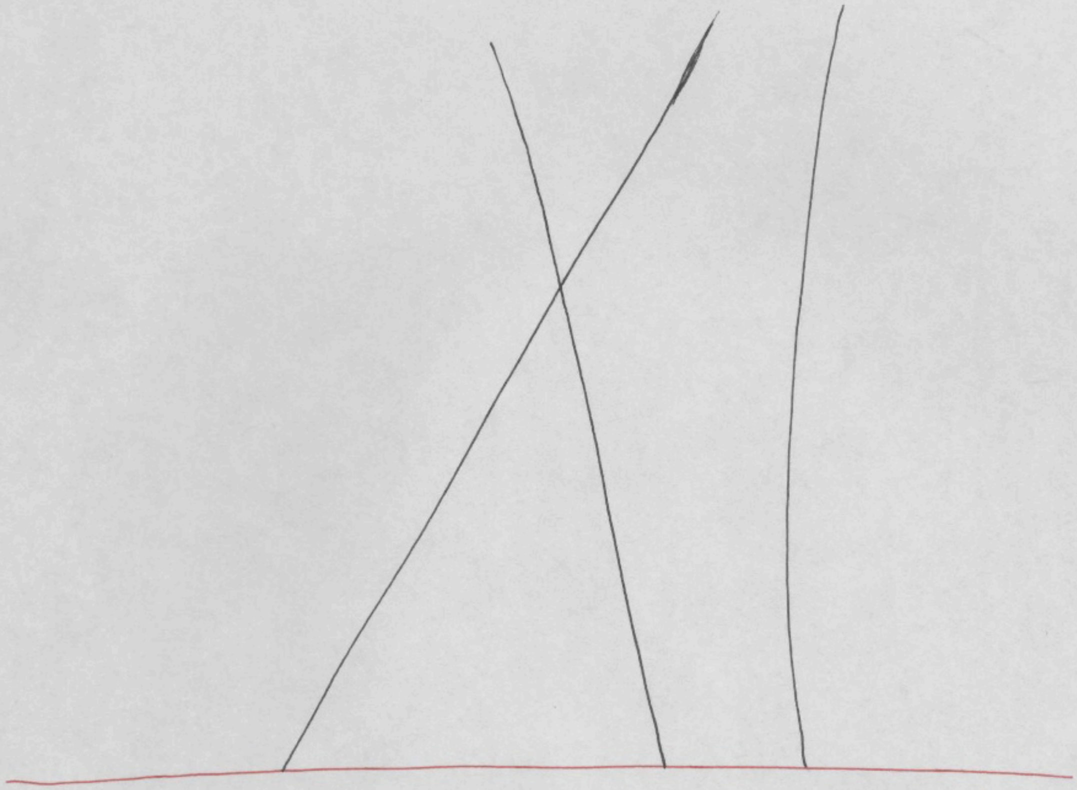
Case 1

For every finite geodesic $g$, $c$ is not bounded by $g$. Let $\tilde{G} = \cap_{g finite} U(g)$, let $\tilde{\mathbb{R}}$ be the broken-line path bounding $\tilde{G}$ from below. (This path consists of all segments $s$ of $\mathbb{R}$ or of finite geodesics such that $s$ is not bounded by any finite geodesic.)

Observe that any connection through $G$ can be written as the direct sum of a finite collection of connections through $\tilde{G}$ and connections through the region below $\tilde{\mathbb{R}}$. Since removing an edge above $\tilde{\mathbb{R}}$ cannot break any connections below it, combing out $c$ can break a connection through $G$ only if it breaks a connection through $\tilde{G}$.

Now $\tilde{G}$ is a pseudocritical, connected graph with compact medial cells and no reentrant geodesics. Hence, $c$ can be combed out in such a way that $\tilde{G}$ remains well-connected. So no connection is broken, and hence the edge through $c$ was inessential.





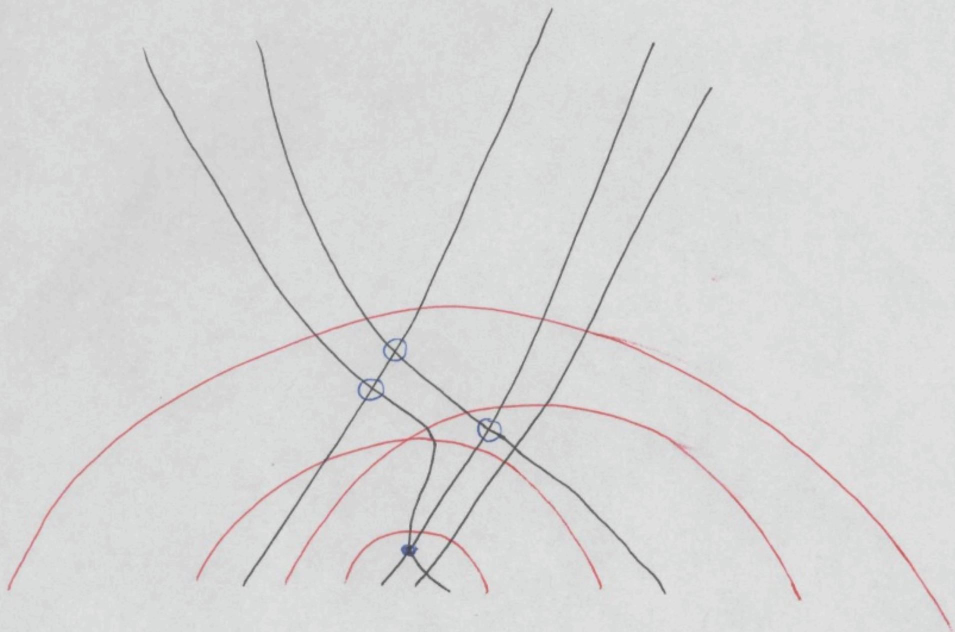
Case 2

$c$ is bounded by a finite collection $C$ of finite geodesics. Construct a set $X$ as follows: (i) $c \in X$, and (ii) given any geodesic $g$ involved in some crossing $x \in X$, if $g$ crosses another infinite geodesic at some crossing $y$ such that $y$ is above $x$ and bounded by a finite geodesic in $C$, then $y \in X$.

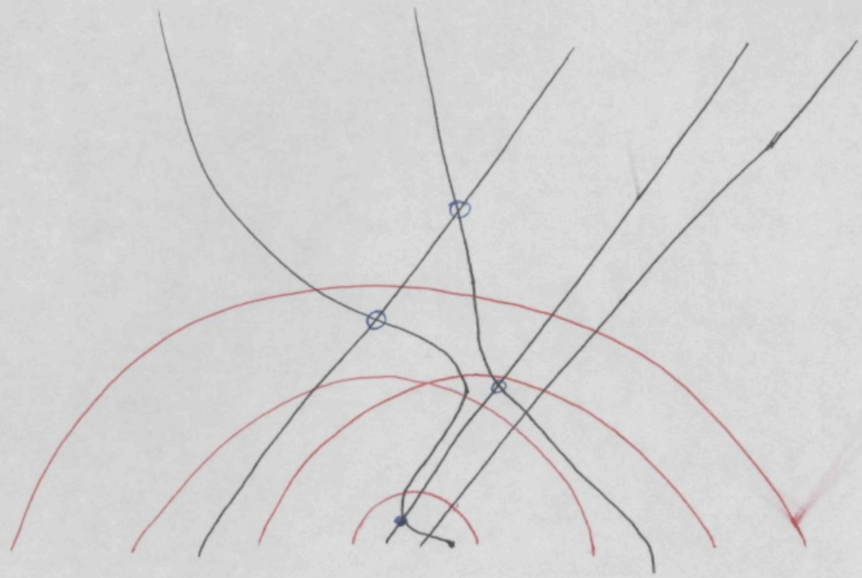
This set is clearly finite, because the region bounded by the geodesics in $C$ is compact. Furthermore, no crossing in $X$ is bounded by any finite geodesic not in $C$, since clearly if an infinite geodesic $g$ has crossings $x$ and $y$, with $y$ above $x$, then every finite geodesic bounding $y$ must also bound $x$.

Hence, by performing a finite sequence of $Y - \Delta$ transformations, each of which moves a crossing $x \in X$ from the bounded to the unbounded portion of some finite geodesic $g \in C$, we obtain a graph $'G$ electrically equivalent to $G$ in which $c$ is not bounded by any finite geodesic. Combing out all crossings in $X$ from $'G$ leaves us with a third graph electrically equivalent to $'G$ by the argument in Case 1.

But this graph is identical to the one obtained by combing out all crossings in $X$ from $G$. It follows that $c$ can be removed from $G$ without breaking any connections.

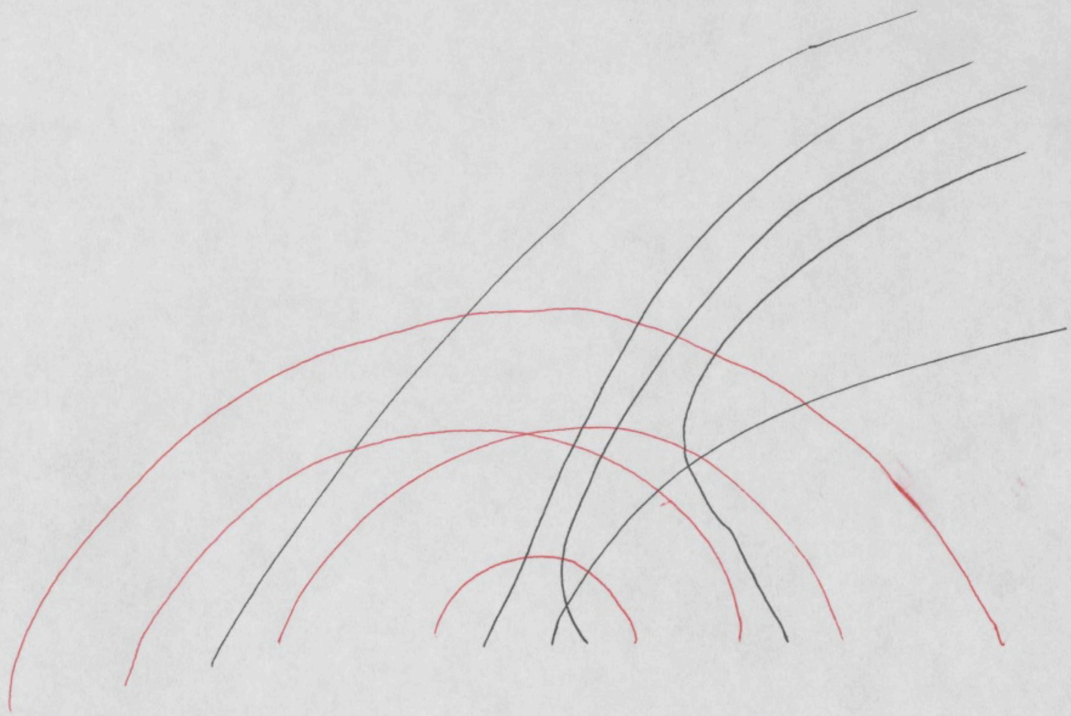


Place a partial ordering on the infinite-infinite crossings bounded by the set of finite geodesics bounding the chosen crossing, and mark off the ones that are above the chosen crossing.



Move each of the marked crossings outside the region bounded by this collection of finite geodesics, respecting the partial ordering. This produces a sequence of electrically equivalent networks.

A network electrically equivalent to the original in which the chosen crossing is not bounded by any finite geodesics.



Uncrossing geodesics that cross above/outside all finite geodesics does not break any connections.

But this produces the same graph as uncrossing the marked crossings without performing the Y-Δ transformations to bring them into the unbounded portion of the graph.

Case 3

$c$ is bounded by infinitely many finite geodesics. Choose any connection $\chi$ through $G$, and fix a set of paths. Then there is a compact region $D$ bounded by the union of a finite collection $C$ of finite geodesics, such that $D$ contains all paths used in $\chi$ and every cell neighboring a path used in $\chi$.

If $c$ is not contained in $D$, then it can be removed without breaking $\chi$.

Suppose $c$ is contained in $D$.

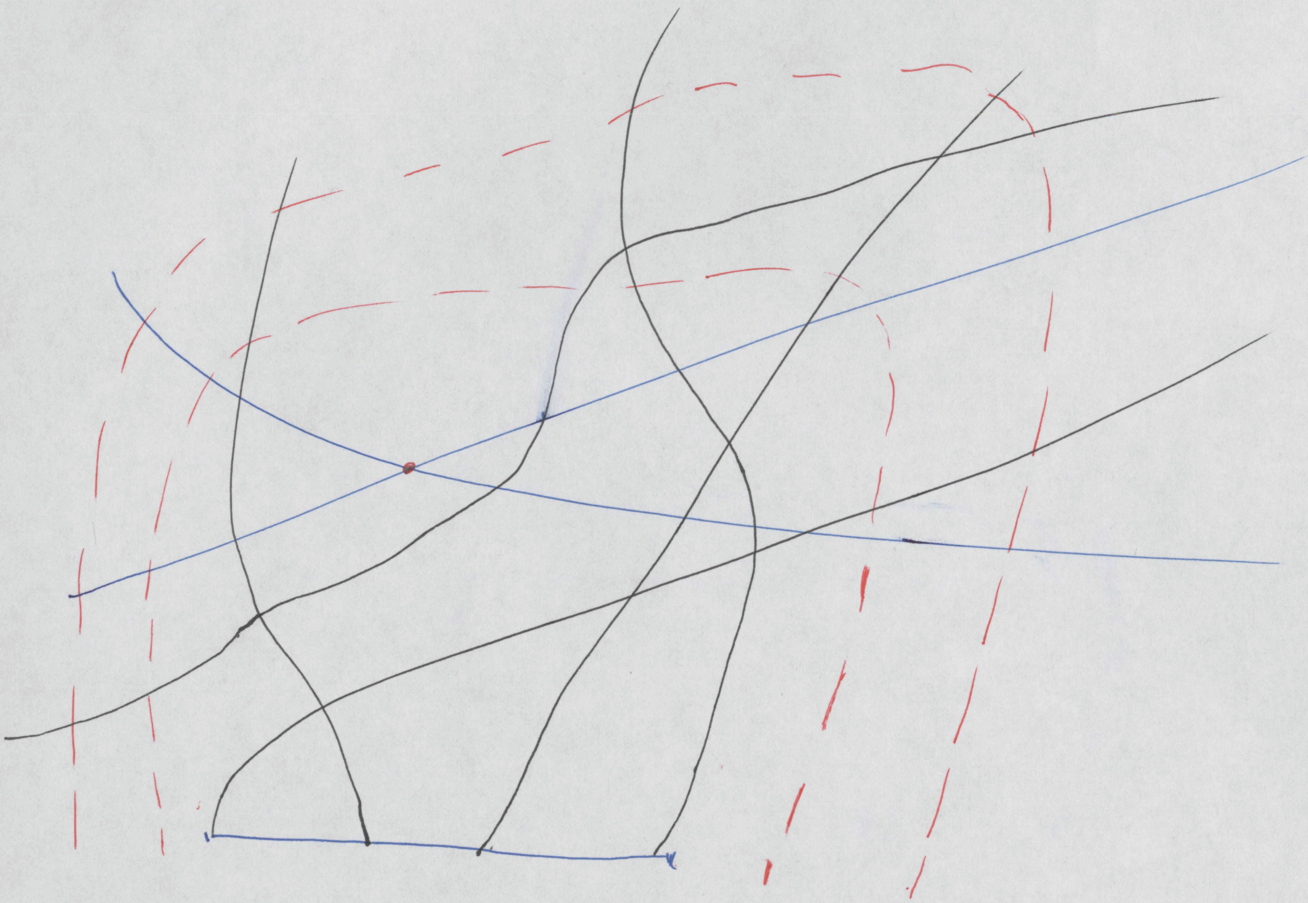
*Remark* 2.10. Observe that a $Y$ is bounded by some finite geodesic $g$ if and only if its equivalent $\Delta$ is bounded by $g$.

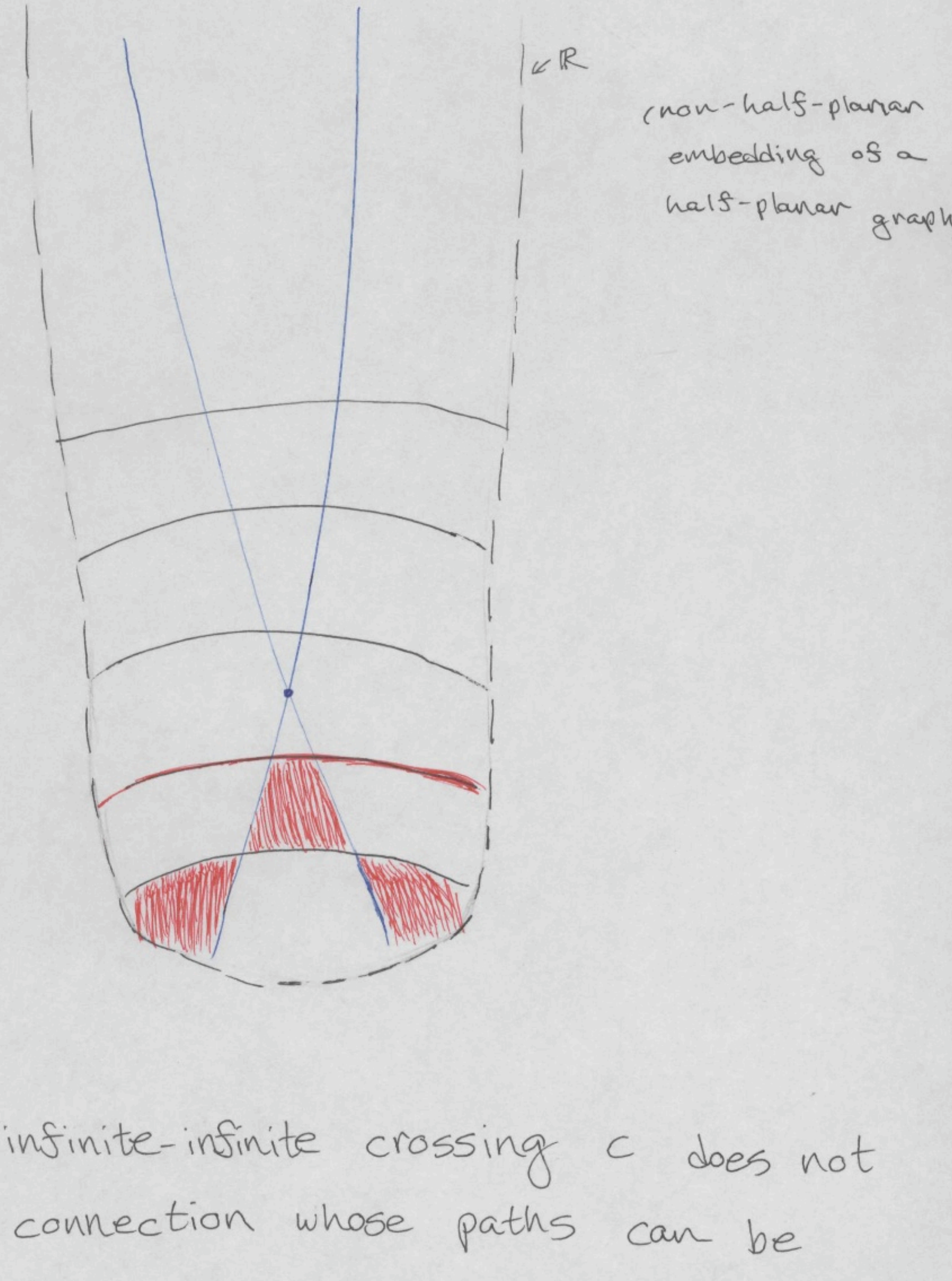
Use a finite sequence of $Y-\Delta$ transformations to move all infinite-infinite crossings out of $D$. Since $D$ fully contained all $Y$s and $\Delta$s involved in the connection, this yields a graph electrically equivalent to $G$ in which all paths used in $\chi$ occur in $D$. However, in this transformed graph, there are no infinite-infinite crossings in $D$. Hence, $\chi$ must also exist in $D$ under the transformation that combs out all infinite-infinite crossings in $D$, including $c$. Since $\chi$ was an arbitrary connection, this shows that no connection is broken by combing out $c$. Hence, the edge through $c$ must be inessential. □

Suppose we have a graph with compact medial cells and no finite geodesics, and that the network is pseudocritical, Suppose we wish to remove a crossing c such that at least one of the geodesics forming it is a line geodesic.

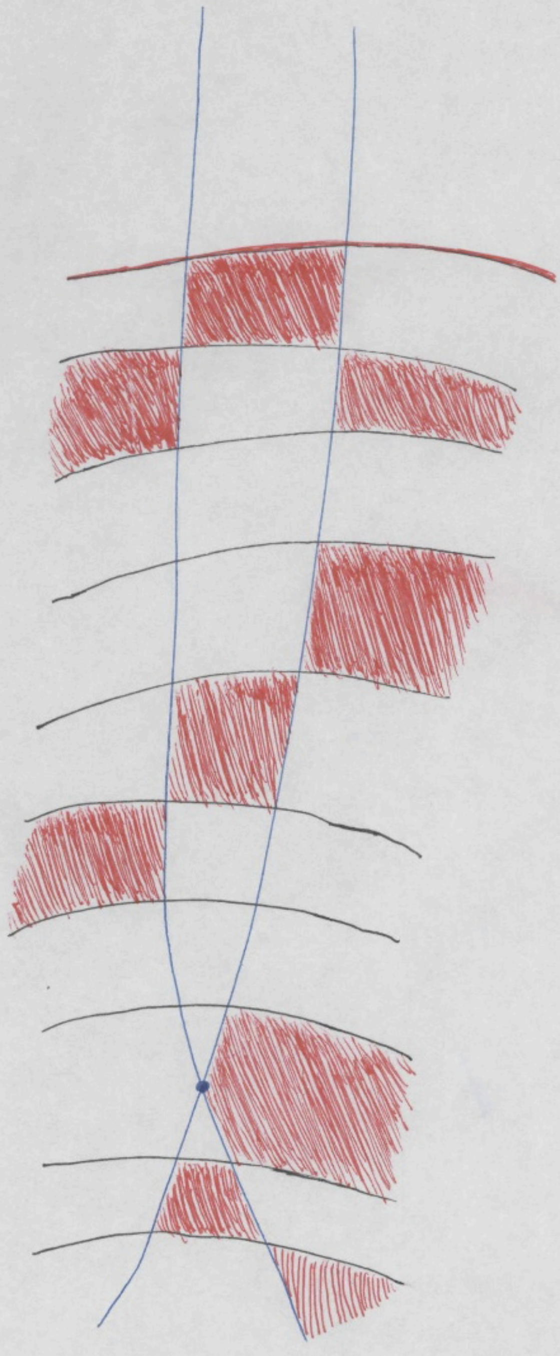
Choose any finite real interval I.

There is a compact region bounding all crossings between pairs of geodesics in the set {g | g is a ray geodesic with an endpoint in I or g is one of the geodesics forming c}.

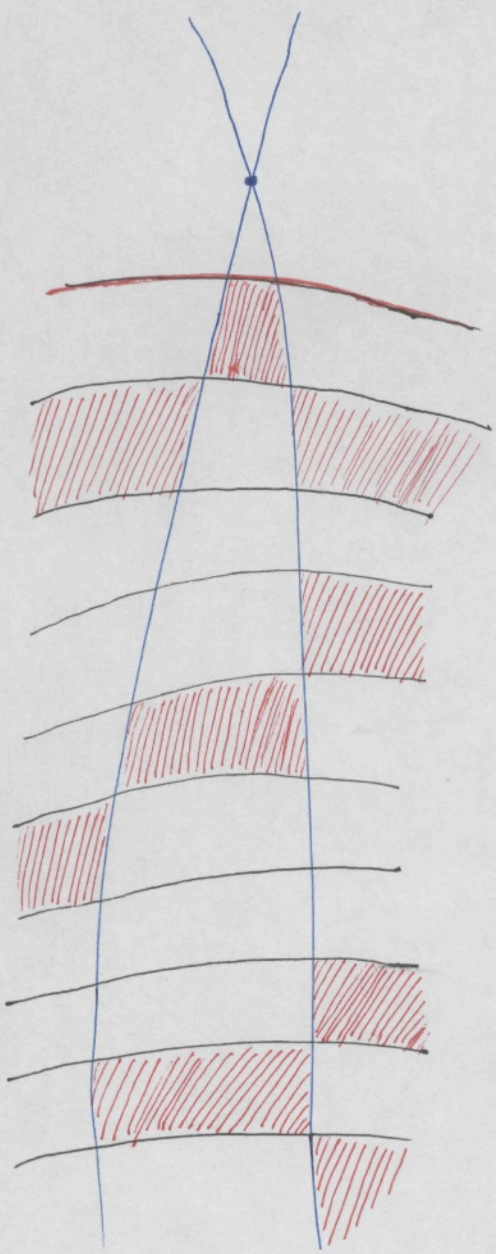




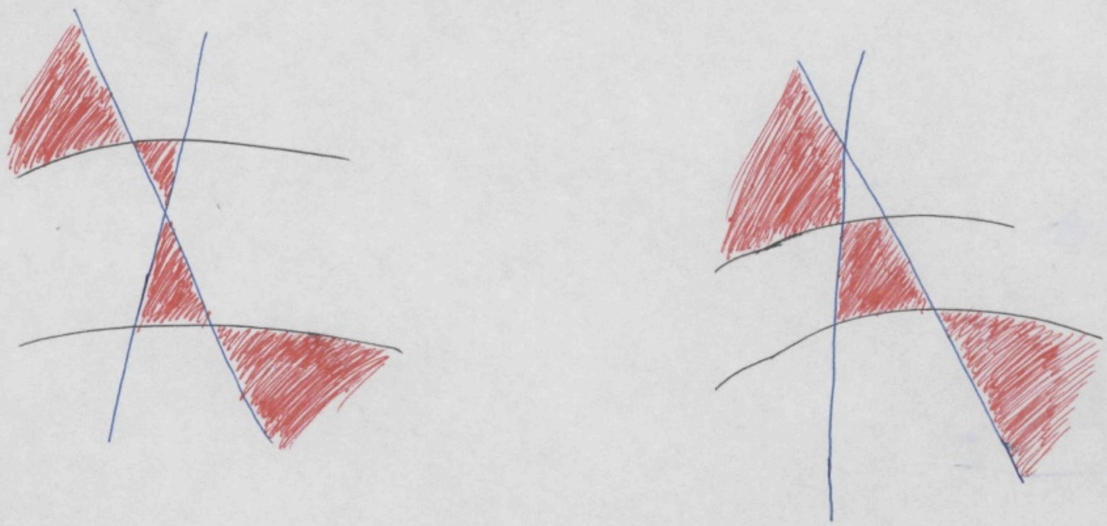

Removing an infinite-infinite crossing c does not break any connection whose paths can be bounded by a collection of finite geodesics that do not bound c.



Suppose that the chosen infinite-infinite crossing is contained in the region bounded by the collection of finite geodesics bounding the paths in the connection.

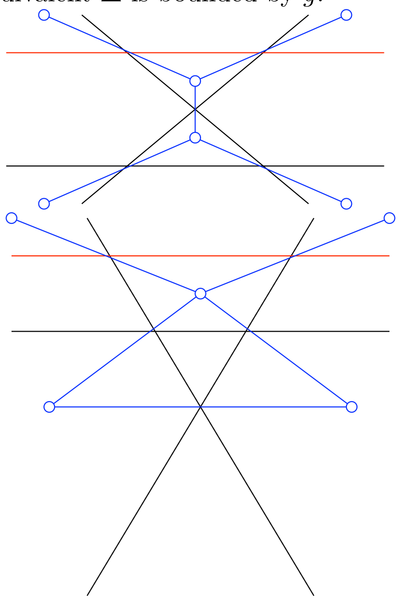


Move the crossing outside the bounded area (along with any infinite-infinite crossings above it and in the bounded region).

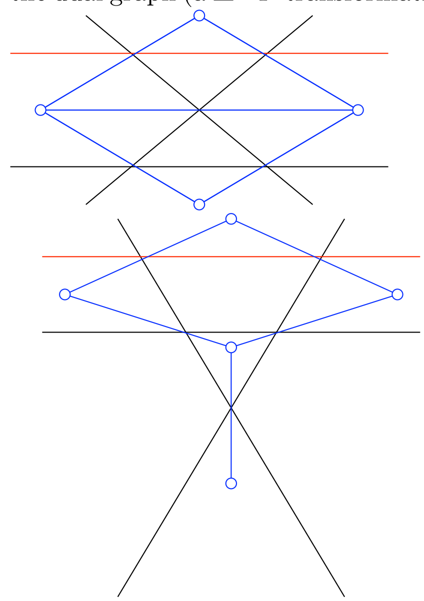


A fixed set of paths that form a $k$-connection between sets $P$ and $Q$ of boundary vertices is preserved under a Y-Δ transformation: that is, applying the Y-Δ transformation to the paths gives a $k$-connection between $P$ and $Q$ in the transformed graph.

Note: A Y is bounded by a finite geodesic $g$ (shown in red) if and only its equivalent $\Delta$ is bounded by $g$.



Effects of moving the same crossing in the dual graph (a $\Delta - Y$ transformation).



**Acknowledgments.** Special thanks to Ian Zemke for developing and explaining the material on which this paper is based, correcting several misconceptions, having innumerable conversations with me about the ideas presented here, and proving a number of conjectures that made my proofs much easier. I am also grateful to Prof. Jim Morrow for organizing the REU that made this research possible and for offering valuable instruction, guidance and feedback, to Peter Mannisto for helpful discussions, and to the University of Washington for supporting this project.

## References

[1] Edward B. Curtis and James A. Morrow. Inverse Problems for Electrical Networks. World Scientific Publishing Co. Pte. Ltd. 2000.
[2] Ian Zemke. Infinite Electrical Networks: Forward and Inverse Problems. 2012.