SOLVABILITY/NONSOLVABILITY OF LINEAR PARTIAL DIFFERENTIAL OPERATORS

NICK REICHERT

Contents

1. Introduction

The following paper discusses several results about solvability of partial differential equations (PDE). It begins with with a statement and proof of the Cauchy-Kowalevski Theorem. Next, using Lewy's example, I show that the theorem does not generalize from the analytic case to the smooth case. Conditions for solvability of more general PDE are given, with a characterization of solvable PDE in a specific case. Finally, I give an overview of more recent work developing the theory further.

I am gratefully indebted to Professor James Morrow for his numerous suggestions and corrections, and for the time he dedicated to discussion of the paper. His help was invaluable.

2. Cauchy-Kowalevski Theorem

The Cauchy-Kowalevski Theorem, 2.3, asserts that under certain conditions, we have existence and uniqueness for solutions of partial differential equations. However, the theorem is somewhat restrictive as its hypotheses make certain assumptions about analyticity. The following proof follows the discussion in [5].

In the following discussion we shall order the set of multi-indices by decreeing that $\alpha < \beta$ if $|\alpha| < |\beta|$ or if $|\alpha| = |\beta|$ and $\alpha_i < \beta_i$, where i is the largest number with $\alpha_i \neq \beta_i$. We shall also use the following elementary result.

Proposition 2.1. *Suppose* $f(x) = \sum_\alpha a_\alpha (x - x_0)^\alpha$ *is convergent near* $x = x_0 \in \mathbb{R}^n$*. Also assume* $g(\xi) = \sum_\beta b_\beta (\xi - \xi^0)^\beta$ *where* $\xi \in \mathbb{R}^m$*,* $b_\beta \in \mathbb{R}^n$*, and* $g(\xi_0) = b_0 = x_0$*. Then* $f(g(\xi)) = \sum_\gamma c_\gamma (\xi - \xi_0)^\gamma$ *is analytic at* ξ_0*, where* $c_\gamma = P_\gamma(\{a_\alpha\}, \{b_\beta\})$ *and* P_γ *is a polynomial such that*

(i) P_γ *is independent of* f *and* g*.*
(ii) P_γ *is a polynomial in the* a_α *and* b_β *for which* $\alpha_j \leq \gamma_j$ *and* $\beta_j \leq \gamma_j$*, all* j*.*
(iii) P_γ *has only non-negative coefficients.*

Proof. Exercise. □

Theorem 2.2. *Suppose* B *is an analytic* $\mathbb{R}^N$*-valued function,* $A_1, \cdots, A_{n-1}$ *are analytic* $N \times N$*-real-matrix-valued functions, and* $\Phi(x)$ *is analytic* $\mathbb{R}^N$*-valued function, each analytic an a neighborhood of the origin of their respective domains. Then there is a neighborhood of the origin in* $\mathbb{R}^n$ *on which there exists a unique analytic function* $Y : \mathbb{R}^n \to \mathbb{R}^N$ *which solves the Cauchy problem*

$$\partial_t Y = \sum_{i=1}^{n-1} A_i(x,t,Y)\partial_{x_i} Y + B(x,t,Y) \tag{1}$$
$$Y(x,0) = \Phi(x)$$

Proof. First consider the case when the A_i and B are independent of t and $\Phi(x) = 0$.

$$\partial_t Y = \sum_{i=1}^{n-1} A_i(x,Y)\partial_{x_i} Y + B(x,Y) \tag{2}$$
$$Y(x,0) = 0$$

Let $Y = (y_1, \cdots, y_N)$, $B = (b_1, \cdots, b_N)$, $A_i = (a^i_{ml})^N_{m,l=1}$. We wish to find

$$y_m = \sum_{\alpha,j} c_m^{\alpha j} x^\alpha t^j \tag{3}$$

for $1 \leq m \leq N$, satisfying (2). The initial condition forces that $c_m^{\alpha 0} = 0$ for all α, m. We have

$$\partial_t y_m = \sum_{i,l} a^i_{ml}(x, y_1, \cdots, y_N)\partial_{x_i} y_l + b_m(x, y_1, \cdots, y_N) \tag{4}$$

Now, we can use the series for the y_k in place of the variables y_k as parameters for a^i_{ml} and b_m. By Proposition 2.1, and using (3) in (4), we rewrite (4) as

$$\sum_{\alpha,j} (j+1) c_m^{\alpha(j+1)} x^\alpha t^j = \sum_{\alpha,j} P_m^{\alpha j}((c_k^{\beta l})_{l \leq j}, d_i) x^\alpha t^j$$

where d_i is the coefficient of A_i and B, and $P_m^{\alpha j}$ is a polynomial with non-negative coefficients. So by uniqueness of power series expansions

$$c_m^{\alpha(j+1)} = \frac{1}{j+1} P_m^{\alpha j}((c_k^{\beta l})_{l \leq j}, d_i)$$

Thus if $c_m^{\alpha l}$ is known for all $l < j$, then $c_m^{\alpha j}$ can be determined. In particular, we find that $c_m^{\alpha j} = Q_m^{\alpha j}(d_i)$, where $Q_m^{\alpha j}$ is a polynomial with non-negative coefficients. This establishes uniqueness.

It remains to show that the series (3) for y_m is valid on a neighborhood of the origin. Suppose that in equations (2) A_i and B are replaced with $\tilde{A}_i$ and $\tilde{B}$, and it is known that an analytic solution $\tilde{Y}$ exists on a neighborhood of the origin. Also assume that the series for $\tilde{A}_i$ and $\tilde{B}$ majorize those of A_i and B. The above formula (3) gives $\tilde{y}_m = \sum_{\alpha,j} \tilde{c}_m^{\alpha j} x^\alpha t^j$, where $\tilde{c}_m^{\alpha j} = Q_m^{\alpha j}(\tilde{d}_i)$ and $Q_m^{\alpha j}$ is the same polynomial as above. As $Q_{\alpha j}$ has non-negative coefficients, $|c_m^{\alpha j}| \le \tilde{c}_m^{\alpha j}$. So the series for $\tilde{Y}$ majorizes the series for Y, and thus the series for Y is valid on some neighborhood of the origin. Hence it suffices to find such an $\tilde{A}_i$ and $\tilde{B}$.

Suppose $\sum_\alpha a_\alpha x^\alpha$ converges on the hypercube $\{x : \max\{|x_j|\} < R\}$. Then let $0 < r < R$, and $x = (r, \cdots, r)$. Then $\sum_\alpha a_\alpha r^{|\alpha|}$ converges, so there is a constant M such that $|a_\alpha r^{|\alpha|}| \le M$ for all α. Thus $|a_\alpha| \le \frac{M}{r^\alpha} \le \frac{M|\alpha|!}{\alpha! r^{|\alpha|}}$. As the n-dimensional geometric series expansion is given by

$$\frac{M}{r - (x_1 + \cdots x_n)} = M \sum_{k=0}^{\infty} \frac{(x_1 + \cdots + x_n)^k}{r^k} = M \sum_{|\alpha| \ge 0} \frac{|\alpha|!}{\alpha! r^{|\alpha|}} x^\alpha$$

we have found a geometric series which majorizes $\sum_\alpha a_\alpha r^{|\alpha|}$. More specifically, if $M > 0$ is large and $r > 0$ is small, then the series for A_i and B are both majorized by the series for

$$\frac{Mr}{r - (x_1 + \cdots + x_{n-1}) - (y_1 + \cdots + y_N)}$$

So consider the Cauchy problem

$$\begin{aligned} \partial_t y_m &= \frac{Mr}{r - \sum_j x_j - \sum_j y_j} \Big(\sum_i \sum_j \partial_{x_i} y_j + 1\Big) \\ y_m(x, 0) &= 0 \end{aligned} \tag{5}$$

First we find a solution u_0 in the simple case

$$\begin{aligned} \partial_t u &= \frac{Mr}{r - s - Nu} (N(n-1)\partial_s u + 1) \\ u(s, 0) &= 0 \end{aligned}$$

where u is a scalar unknown in the two variables s and t. This can be rewritten as

$$(r - s - Nu)\partial_t u - MrN(n-1)\partial_s u = Mr$$

Using elementary PDE theory (see [5]), we obtain

$$u(s,t) = \frac{r - s - \sqrt{(r-s)^2 - 2MrNnt}}{Mn}$$

In the more general case of (5), let $y_m(x,t) = u(x_1 + \cdots + x_{n-1}, t)$, $1 \le m \le N$. Then the system (5) is satisfied.

Now consider the case of (1) where the A_i and B may depend on t and Φ may be nonzero. If $U(x,t) = Y(x,t) - \Phi(x)$, then Y satisfies (1) if and only if U satisfies the system

$$\begin{aligned} \partial_t U &= \sum_{i=1}^{n-1} \tilde{A}_i(x, t, U)\partial_{x_i} U + \tilde{B}(x, t, U) \\ U(x, 0) &= 0 \end{aligned}$$

So we can assume $\Phi \equiv 0$. Next, let

$$V(x,t) = (u_0(x,t), U(x,t)) = (u_0(x,t), u_1(x,t), \cdots, u_N(x,t))$$

where $\partial_t u_0(x,t) = 1$ and $u_0(x,0) = 0$. Hence $u_0 \equiv t$, so in equations (1) we can replace t by u_0 in $\tilde{A}_i$ and $\tilde{B}$ by adding the extra equation and the extra initial condition. Thus the proof of existence in the general case (1) is complete. As analytic functions are completely determined by the values of their derivatives at a single point, an analytic solution to (1) is necessarily unique. □

We are now prepared to prove the classical result.

Corollary 2.3. (Cauchy-Kowalevski Theorem) *Suppose $F, \phi_0, \cdots, \phi_{k-1}$ are analytic near the origin, and S is an analytic hypersurface containing the origin. Assume that the equation $F = 0$ can be solved for $\partial_t^k u$ to obtain ∂_t^k as a function G of the remaining variables. Then there is a neighborhood of the origin on which the Cauchy problem*

$$0 = F(x, (\partial^\alpha)_{|\alpha|\leq k}) \tag{6}$$
$$\partial_\nu^j u = \phi_j \text{ on } S, 0 \leq j < k$$

has a unique analytic solution.

Proof. We can make an analytic change of coordinates so that some neighborhood of the origin in S is mapped to the hyperplane $t = 0$. So we can assume the system (6) is of the form

$$\partial_t^k u = G(x, t, (\partial_x^\alpha \partial_t^j u)_{|\alpha|+j\leq k, j<k}) \tag{7}$$
$$\partial_t^j u(x, 0) = \phi_j(x), 0 \leq j < k$$

Now consider the system of equations and initial conditions

$$\partial_t y_{\alpha j} = y_{\alpha(j+1)}, |\alpha| + j < k \tag{8}$$
$$\partial_t y_{\alpha j} = \partial_{x_i} y_{(\alpha-1_i)(j+1)}, |\alpha| + j = k, j < k \tag{9}$$
$$\partial_t y_{0k} = \frac{\partial G}{\partial t} + \sum_{|\alpha|+j<k} \frac{\partial G}{\partial y_{\alpha j}} y_{\alpha(j+1)} \tag{10}$$
$$y_{\alpha j}(x, 0) = \partial_x^\alpha \phi_j(x), j < k \tag{11}$$
$$y_{0k}(x, 0) = G(x, 0, (\partial_x^\alpha \phi_j(x))_{|\alpha|+j\leq k, j<k}) \tag{12}$$

If $Y = (y_1, \cdots, y_k)$, then by Theorem 2.2 the system (8)-(12) has a unique analytic solution near zero. Hence it suffices to show that $u = y_{00}$ satisfies (7). Now, equation (8) implies

$$y_{\alpha(j+1)} = \partial_t^l y_{\alpha j}, \qquad j + l \leq k \tag{13}$$

Combining this with equation (9) gives

$$\partial_t y_{\alpha j} = \partial_t \partial_{x_i} y_{(\alpha-1_i)j}$$

and so

$$y_{\alpha j}(x, t) = \partial_{x_i} y_{(\alpha-1_i)j}(x, t) + c_{\alpha j}(x)$$

for some $c_{\alpha j}$. However, by equation (11),

$$y_{\alpha j}(x, 0) = \partial_x^\alpha \phi_j(x) = \partial_{x_i} \partial_x^{\alpha-1_i} \phi_j(x) = \partial_{x_i} y_{(\alpha-1_i)j}(x, 0)$$

and $c_{\alpha j} = 0$. Hence

$$y_{\alpha j} = \partial_{x_i} y_{(\alpha-1_i)j}, \qquad |\alpha| + j = k, j < k \tag{14}$$

Now, by (10), (13), and (14),

$$\partial_t y_{0k} = \frac{\partial G}{\partial t} + \sum_{|\alpha|+j\leq k, j<k} \frac{\partial G}{\partial y_{\alpha j}}\frac{\partial y_{\alpha j}}{\partial t} = \frac{\partial}{\partial t}(G(x,t,(y_{\alpha j})))$$

Thus

$$y_{0k}(x,t) = G(x,t,(y_{\alpha j}(x,t))) + c_{0k}(x)$$

for some c_{0k}. However, equations (11), (12) imply

$$y_{0k}(x,0) = G(x,0,(\partial_x^\alpha(\phi_j(x))) = G(x,0,(y_{\alpha j}(x,0)))$$

so that $c_{0k} = 0$ and

$$y_{0k} = G(x,t,(y_{\alpha j})_{|\alpha|+j\leq k, j<k}) \tag{15}$$

Next we show by induction on $k - j - |\alpha|$ that

$$\partial_{\alpha j} = \partial_{x_i} y_{(\alpha - 1_i)j}, \alpha \neq 0$$

The base case $k = j + |\alpha|$ is shown in (14). By (8) and (13),

$$\partial_t y_{\alpha j} = y_{\alpha(j+1)} = \partial_{x_i} y_{(\alpha-1_i)(j+1)} = \partial_t \partial_{x_i} y_{(\alpha-1_i)j}$$

and so

$$y_{\alpha j}(x,t) = \partial_{x_i} y_{(\alpha-1_i)j}(x,t) + c_{\alpha j}(x)$$

Equation (11) gives

$$\partial_{\alpha j}(x,0) = \partial_x^\alpha \phi_j(x) = \partial_{x_i}\partial_x^{\alpha-1_i}\phi_j(x) = \partial_{x_i} y_{(\alpha-1_i)j}(x,0)$$

so that $c_{\alpha j} = 0$ and the induction is complete.

Finally, (13) and (14) give

$$y_{\alpha j} = \partial_x^\alpha \partial_t^j y_{00} \tag{16}$$

By (11), (15), and (16), $u = y_{00}$ is a solution to (7). □

Note that in the above discussion, it was assumed that all functions were real-valued. By considering $\mathbb{C}^n$-valued functions as $\mathbb{R}^{2N}$-valued functions, we need not assume that the functions are real-valued.

3. Lewy's Counterexample

One might naturally assume that the Cauchy-Kowalevski theorem would extend to smooth partial differential equations. In 1957, Hans Lewy [9] showed that this was not the case. The following exposition derives from and expands upon on Lewy's paper and the discussion of the result in [4], [5], and [6].

Let L be the differential operator defined on $\mathbb{R}^3 = \{(x,y,t)\}$ by

$$L = \partial_x + i\partial_y - 2i(x+iy)\partial_t \tag{17}$$

Lemma 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If there exists a C^1 function $u(x,y,t)$ such that $Lu = f(t + 2y_0x - 2x_0y)$ on a neighborhood U of (x_0,y_0,t_0), then f is analytic at $t = t_0$.*

Proof. First assume that $x_0 = y_0 = 0$. Let $R > 0$ be such that $\{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 < R, |t - t_0| < R\} \subseteq U$. Let $z = x + iy = re^{i\theta}$, and let $s = r^2$. Define $V(t, r)$ for $0 < r < R$ and $|t - t_0| < R$ by the contour integral

$$V = \int_{|z|=r} u(x, y, t)dz = ir \int_0^{2\pi} u(r\cos\theta, r\sin\theta, t)e^{i\theta}d\theta$$

Then by Green's Theorem,

$$\begin{aligned} V &= i \iint_{|z|\le r} (\partial_x u + i\partial_y u)(x, y, t)dxdy \\ &= i \int_0^r \int_0^{2\pi} (\partial_x u + i\partial_y u)(\rho\cos\theta, \rho\sin\theta, t)\rho d\theta d\rho \end{aligned}$$

Thus

$$\begin{aligned} \partial_r V &= i \int_0^{2\pi} (\partial_x u + i\partial_y u)(r\cos\theta, r\sin\theta, t)rd\theta \\ &= \int_{|z|=r} (\partial_x u + i\partial_y u)(x, y, t)r\frac{dz}{z} \end{aligned}$$

Since $Lu = f$, we have

$$\begin{aligned} \partial_s V &= \frac{1}{2r}\partial_r V \\ &= \int_{|z|=r} (\partial_x u + i\partial_y u)(x, y, t)\frac{dz}{2z} \\ &= i\int_{|z|=r} \partial_t u(x, y, t)dz + \int_{|z|=r} f(t)\frac{dz}{2z} \\ &= i\partial_t V + \pi i f(t) \end{aligned}$$

Let $F(t) = \int_{t_0}^t f(\alpha)d\alpha$, and $U(t, s) = V(t, s) + \pi F(t)$. Then $\partial_t U + i\partial_s U = 0$, i.e. U satisfies the Cauchy Riemann equations. Thus U is a holomorphic function of $w = t + is$ in the region $0 < s < R^2$, $|t - t_0| < R$, and U is continuous up to the line $s = 0$. Since $V = 0$ when $s = 0$, $U(0, t) = \pi F(t)$ is real-valued. By the reflection principle, $U(t, -s) := \bar{U}(t, s)$ defines an analytic continuation of U to a neighborhood of the origin. Hence $U(t, 0) = \pi F(t)$ is analytic near t_0, and $f = F'$ is as well. This completes the argument in the case $x_0 = y_0 = 0$.

Now suppose x_0 and y_0 are arbitrary, and u satisfies the hypotheses of the lemma. In particular,

$$Lu(x, y, t) = f(t + 2y_0x - 2x_0y)$$

near (x_0, y_0, t_0), and $u \in C^1$ near (x_0, y_0, t_0). Define $\hat{u}(x, y, t) = u(x+x_0, y+y_0, t-2y_0x + 2x_0y)$. Then $\hat{u} \in C^1$ near $(0, 0, t_0)$, and by the chain rule

$$\begin{aligned} L\hat{u}(x, y, t) &= L(u(x + x_0, y + y_0, t - 2y_0x + 2x_0y)) \\ &= (\partial_x u)(x + x_0, y + y_0, t - 2y_0x + 2x_0y)- \\ &\quad 2y_0(\partial_t u)(x + x_0, y + y_0, t - 2y_0x + 2x_0y)+ \\ &\quad i(\partial_y u)(x + x_0, y + y_0, t - 2y_0x + 2x_0y)+ \\ &\quad 2ix_0(\partial_t u)(x + x_0, y + y_0, t - 2y_0x + 2x_0y)+ \\ &\quad 2i(x + iy)(\partial_t u)(x + x_0, y + y_0, t - 2y_0x + 2x_0y) \\ &= (\partial_x u)(x + x_0, y + y_0, t - 2y_0x + 2x_0y)+ \\ &\quad (\partial_y u)(x + x_0, y + y_0, t - 2y_0x + 2x_0y)+ \\ &\quad 2i((x + x_0) + i(y + y_0))(\partial_t u)(x + x_0, y + y_0, t - 2y_0x + 2x_0y) \\ &= f((t - 2y_0x + 2x_0y) + 2y_0x - 2x_0y) \\ &= f(t) \end{aligned}$$

near $(0, 0, t_0)$. Thus by the earlier argument, $f(t)$ is analytic at t_0. □

Put another way, if f is not analytic at $t = t_0$, there is no C^1 function $u(x, y, t)$ for which $Lu = f$ on any neighborhood of (x_0, y_0, t_0)–even if f is smooth!

Next we prove the existence of smooth, periodic functions on $\mathbb{R}$ which are nowhere analytic. The result can be shown in many ways. See [6] and [10] for examples arising from trigonometric series. [3] uses a Baire category argument to show that "most" smooth functions are nowhere analytic (in the same sense that "most" continuous functions are nowhere differentiable). The exposition given here is based on [8].

Lemma 3.2. *There exists periodic $\psi \in C^\infty(\mathbb{R})$ which is not analytic at any point.*

Proof. Let $\alpha(x) = \begin{cases} 0 & \text{if } x \le 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$. Then it is well known that α is smooth. Let $\beta(x) = \alpha(x)\alpha(1-x)$. Finally, let $\gamma_j(x) = \frac{1}{j!}\beta(2^jx - \lfloor 2^jx \rfloor)$ and $\gamma(x) = \sum_{j=1}^\infty \gamma_j(x)$. Each γ_j is smooth as all the derivatives of β vanish at 0 and 1. Moreover, γ is periodic. γ is also smooth as $\sum_{j=0}^\infty \gamma_j^{(i)}(x)$ converges uniformly for each i. Now suppose that γ is analytic at some point x. Since analyticity at a point implies analyticity on a neighborhood of that point, γ is analytic at some dyadic rational $r = p/2^k$ with p odd. $\gamma_j(x)$ is analytic at r for $1 \le j \le k-1$, so $\tilde{\gamma}(x) := \sum_{j=k}^\infty \gamma_j(x)$ is analytic at r. However, $\tilde{\gamma}^{(i)}(r) = 0$ for all $i \ge 0$, and $\tilde{\gamma}(x) > 0$ on any small punctured neighborhood of x. This is a contradiction. Hence γ is nowhere analytic. □

We will now construct a function f for which $Lu = f$ has no solutions at any point.

Lemma 3.3. *Let ψ be as above, and suppose $Q_j = (x_j, y_j, t_j)$ is an enumeration of $\mathbb{Q}^3$. If $\rho_j = |x_j| + |y_j|$, let $c_j = 2^{-j}e^{-\rho_j}$. Then for any $\epsilon \in l^\infty(\mathbb{R})$, the series $\sum_{j=1}^\infty \epsilon_j c_j \psi(t - 2y_jx + 2x_jy) =: F_\epsilon(x, y, t)$ and all of its formal derivatives converge uniformly. In particular, $F_\epsilon \in C^\infty(\mathbb{R}^3)$.*

Proof. ϕ is periodic, so that $M_k := \sup_{t\in\mathbb{R}} |\psi^{(k)}(t)|$ is finite for all k. Thus for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$\begin{aligned} |D^\alpha \epsilon_j c_j \psi(t - 2y_j x + 2x_j y)| &\leq \|\epsilon\| c_j M_{|\alpha|} 2^{|\alpha|} \rho_j^{|\alpha|} \\ &= 2^{-j+|\alpha|} \|\epsilon\| M_{|\alpha|} \rho_j^{|\alpha|} e^{-\rho_j} \\ &\leq 2^{-j+|\alpha|} \|\epsilon\| M_{|\alpha|} \left(\frac{|\alpha|}{e}\right)^{|\alpha|} \end{aligned} \tag{18}$$

since $\rho_j^{|\alpha|} e^{-\rho_j} \leq \frac{|\alpha|}{e}^{|\alpha|}$ for $\rho_j \geq 0$, by elementary calculus. So we have shown that $|D^\alpha \epsilon_j c_j \psi(t - 2y_j x + 2x_j y)| \leq K_\alpha 2^{-j}$ for some $K_\alpha \in \mathbb{R}$. Hence the series for $D^\alpha F_\epsilon$ converges uniformly, so that $F_\epsilon \in C^\infty(\mathbb{R}^3)$. □

Next we provide a preliminary result for use in a Baire Category argument.

Lemma 3.4. *Let Q_j be as in the above lemma. For $j, n \in \mathbb{N}$, define $\Upsilon_{j,n} = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x} - Q_j| < n^{-1/2}\}$. Let $E_{j,n} \subset l^\infty$ be the collection of ϵ for which a solution $u_\epsilon(x, y, t) \in C^1(\Upsilon_{j,n})$ of $Lu_\epsilon = F_\epsilon(x, y, z)$ exists, with*

(i) $u_\epsilon(Q_j) = 0$

(ii) $|D^\alpha u_\epsilon(P)| \leq n$ *for* $|\alpha| \leq 1$, $P \in \Upsilon_{j,n}$

(iii) $|D^\alpha u_\epsilon(P) - D^\alpha u_\epsilon(Q)| \leq n|P - Q|^{1/n}$ *for* $|\alpha| = 1$, $P, Q \in \Upsilon_{j,n}$

Then each $E_{j,n}$ is a closed, nowhere dense subset of l^∞.

Proof. First I will show that $E_{j,n}$ is closed. Suppose $\epsilon \in l^\infty$ and $\epsilon_1, \epsilon_2, \cdots \in E_{j,n}$ with $\lim_{k\to\infty} \|\epsilon - \epsilon_k\| = 0$. Taking $\alpha = 0$ in equation (18), $|F_\epsilon - F_{\epsilon_k}| = |F_{\epsilon-\epsilon_k}| \leq M_0 \|\epsilon - \epsilon_k\|$. So $F_{\epsilon_k} \to F$. Let u_{ϵ_k} be a solution of $Lu_{\epsilon_k} = F_{\epsilon_k}(x, y, z)$ satisfying the three properties given in the statement of the lemma. Note that the u_{ϵ_k} are equi-bounded and equi-continuous in $\Upsilon_{j,n}$. By the Arzela-Ascoli Theorem, there exists a subsequence of the u_{ϵ_k} which converge uniformly to a function u (and the derivatives converge uniformly). u must satisfy (i)-(iii) and also $Lu = F_\epsilon$, so $\epsilon \in E_{j,n}$. Thus $E_{j,n}$ is closed.

Let c_j as in the statement of Lemma 3.3, and define $\delta = (0, \cdots, 0, 1/c_j, 0, \cdots,)$ be the sequence which is zero except in the jth position. By definition, $F_\delta = \psi(t - 2y_0 x + 2x_0 y)$.

Now suppose ϵ is an interior point of $E_{j,n}$. Then there exists $\theta > 0$ such that $\epsilon' = \epsilon + \theta\delta \in E_{j,n}$. Let u,u' be solutions of $Lu = F_\epsilon$ and $Lu' = F_{\epsilon'}$, respectively, and satisfying properties (i)-(iii). If $u'' = (u' - u)/\theta$, then $u'' \in C^1$ and $Lu'' = F_\delta = \psi$ near Q_j. This contradicts Lemma 3.1, as ψ is nowhere analytic. □

We are now ready to prove the main result.

Theorem 3.5. *Let L be as above as in equation (17). Then there exists $F \in C^\infty(\mathbb{R}^3)$ such that $Lu = F$ has no solution u on any open set $\Upsilon \subset \mathbb{R}^3$ with $u \in C^1(\Upsilon)$ and $\partial_x u, \partial_y u, \partial_t u$ Holder continuous on Υ.*

Proof. Assume for the sake of contradiction that the theorem is false. Then for all $\epsilon \in l^\infty$, there exists an open set Υ_ϵ and a solution u of $Lu = F_\epsilon$ on Υ_ϵ with Holder continuous first derivatives. For some j, $Q_j \in \Upsilon_\epsilon$. So $\Upsilon_{j,n} \subset \Upsilon_\epsilon$ for n large. Also, u will satisfy properties (ii) and (iii) of Lemma 3.4 if n is large enough. Replacing u by $u - u(Q_j)$, we can also assume that u satisfies property (i) as well. Thus $\epsilon \in E_{j,n}$, and $l^\infty = \cup_{j,n} E_{j,n}$. Combining this with Lemma 3.4, we obtain a contradiction to the Baire Category Theorem. □

4. Poisson-Bracket Condition

The development given in this section is based on [14] and [7].

4.1. **Definitions and Background.**

- For convenience, let $\lambda^s(\xi) = (1+|\xi|^2)^{s/2}$, with $\xi \in \mathbb{R}^n$ and $s \in \mathbb{R}$.
- We will often make the association $T^*(\mathbb{R}^n) \cong \mathbb{R}^{2n}$. The variables x and ξ will typically represent points in $\mathbb{R}^n$ and $T_x(\mathbb{R}^n)$, respectively. Moreover, if $f : \mathbb{R}^{2n} \to \mathbb{R}$ or $\mathbb{C}$, then by $\partial_x f$ and $\partial_\xi f$ we mean the x and ξ gradients, $(\partial_1 f, \cdots, \partial_n f)$ and $(\partial_{n+1} f, \cdots, \partial_{2n} f)$, respectively. If α and β are multi-indices, then $\partial_x^\alpha f$ and $\partial_\xi^\beta f$ denote derivatives of f in the x and ξ variables, respectively.
- Suppose $m \in \mathbb{R}$ and $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ (in this paper, C^∞ functions are complex-valued). Then a is said to be a *symbol of order m*, written $a \in S^m$, if each function $\lambda^{|\beta|-m}\partial_x^\alpha \partial_\xi^\beta a$ is bounded on $\mathbb{R}^n \times \mathbb{R}^n$ for all multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$. Note that $l \leq m$ implies $S^l \subset S^m$. Thus we define $S^{-\infty} = \cap_m S^m$ and $S^\infty = \cup_m S^m$. Note that for $a \in S^m$, $b \in S^l$, $\alpha, \beta \in \mathbb{Z}_+^n$, we have $\partial_x^\alpha \partial_\xi^\beta a \in S^{m-|\beta|}$ and $ab \in S^{m+l}$. Occaisionally we will formally substitute the differential operator $D = -i(\partial_1, \cdots, \partial_n)$ for the variable ξ in the expression $a(x,\xi)$. When a is of the form $a(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ formally replacing ξ by D makes sense. Lemma 4.4 makes the general definition precise.
- Let $m \geq 0$. Then we say $a \in A^m$, or a is an *amplitude of order m*, if $a \in C^\infty(\mathbb{R}^n)$ and the functions $(1+|x|^2)^{-m/2}\partial^\alpha a(x)$ are bounded on $\mathbb{R}^n$ for all $\alpha \in \mathbb{Z}_+^n$. On the space A^m, we define the norms

$$|||a|||_k = \max_{|\alpha|\leq k} \|(1+|x|^2)^{-m/2}\partial^\alpha a\|_{L^\infty}$$

The following six lemmas are standard results about symbols and oscillatory integrals. Their proofs appear in the appendix.

Lemma 4.1. *If $a \in S^0$ and $F \in C^\infty(\mathbb{C})$, then $F(a) \in S^0$.*

Lemma 4.2. *Let $a_j \in S^{m-j}$ for $j \in \mathbb{Z}_+$. Then there exists a symbol $a \in S^m$ such that for any $k \in \mathbb{Z}_+$,*

$$a - \sum_{j=1}^{k} a_j \in S^{m-k}$$

Moreover, a is unique modulo $S^{-\infty}$. a can be chosen so that supp $a \subset \cup_j$ supp a_j.

- If a is as in the above lemma, then we write $a \sim \sum_j a_j$, and say that *the $\{a_j\}$ are asymptotic to a.*

Lemma 4.3. *Let q be a nondegenerate real quadratic form on $\mathbb{R}^n$, $a \in A^m$, and $\phi \in \mathcal{S}$ such that $\phi(0) = 1$. Then the limit*

$$\lim_{\epsilon\to 0} \int e^{iq(x)} a(x)\phi(\epsilon x)dx \tag{19}$$

exists and is independent of ϕ. If in addition $a \in L^1$, then the limit is equal to $\int e^{iq(x)}a(x)dx$. Thus we denote the limit (19) as $\int e^{iq(x)}a(x)dx$, regardless of *whether* $a \in L^1$. $\int e^{iq(x)}a(x)dx$ *is said to be an* oscillatory integral. *Also,*

$$|\int e^{iq(x)}a(x)dx| \leq C_{q,m}|\|a\||_{m+n+1}$$

where $C_{q,m}$ *depends only on* q *and* m.

Lemma 4.4. *If* $a \in S^\infty$ *and* $\phi \in \mathcal{S}$, *then*

$$a(x,D)\phi(x) := (2\pi)^{-n}\int e^{i\langle x,\xi\rangle}a(x,\xi)\hat{\phi}(\xi)d\xi$$

defines a function $a(x,D)\phi \in \mathcal{S}$. *Moreover, there exist constants* $N \in \mathbb{Z}_+$ *and* C_k *for* $k \in \mathbb{Z}_+$ *depending on* a *such that* $|a(x,D)\phi|_k \leq C_k|\phi|_{k+N}$.

- If $a \in S^\infty$, then we say that the *pseudodifferential operator of symbol* a is the operator $a(x,D) : \mathcal{S}' \to \mathcal{S}'$ defined by

$$(a(x,D)u,\phi) = (u, a^*(x,D)\phi) \tag{20}$$

  for $u \in \mathcal{S}'$, $\phi \in \mathcal{S}$. If $a \in S^m$, then $a(x,D)$ is said to have order m. We define $\Psi^m = \{a(x,D) : a \in S^m\}$, $\Psi^\infty = \cup_m \psi^m$, and $\Psi^{-\infty} = \cap_m \Psi^m$. Elements of $\Psi^{-\infty}$ are called *smoothing operators*.
- Note that pseudo-differential operators generalize linear partial differential operators. In particular, if $a(D)$ is simply a linear partial differential operator, $a(D) = \sum_\alpha a_\alpha D^\alpha$, then equation (20) is a consequence of the Fourier inversion formula.

Lemma 4.5. *Oscillatory integrals are very similar to usual integrals. In particular, they satisfy the following properties:*

(*i*) *Change of Variables: If* $A \in GL_n(\mathbb{R})$, *then*

$$\int e^{iq(Ay)}a(Ay)|\det A|dy = \int e^{iq(x)}a(x)dx$$

(*ii*) *Integration by Parts: If* $a \in A^m$, $b \in A^l$, *and* $\alpha \in \mathbb{Z}^n_+$, *then*

$$\int e^{iq(x)}a(x)\partial^\alpha b(x)dx = \int b(x)(-\partial)^\alpha(e^{iq(x)}a(x))dx$$

(*iii*) *Differentiation Under* $\int$: *If* $a \in A^m(\mathbb{R}^n \times \mathbb{R}^p)$, *then* $\int e^{iq(x)}a(x,y)dx \in A^m(\mathbb{R}^p)$. *Moreover, for all* $\alpha \in \mathbb{Z}^n_+$,

$$\partial^\alpha_y \int e^{iq(x)}a(x,y)dx = \int e^{iq(x)}\partial^\alpha_y a(x,y)dx$$

(*iv*) *Fubini's Theorem: If* $a \in A^m(\mathbb{R}^n \times \mathbb{R}^p)$ *as in* (*iii*) *and if* r *is a nondegenerate real quadratic form on* $\mathbb{R}^p$, *then*

$$\int e^{ir(y)}(\int e^{iq(x)}a(x,y)dx)dy = \int e^{i(q(x)+r(y))}a(x,y)dxdy$$

Lemma 4.6. *Let* $a \in S^m$ *and* $b \in S^l$. *The oscillatory integrals*

$$a^*(x,\xi) = (2\pi)^{-n}\int e^{-i\langle y,\eta\rangle}\bar{a}(x-y,\xi-\eta)dyd\eta$$

$$a\#b(x,\xi) = (2\pi)^{-n}\int e^{-i\langle y,\eta\rangle}a(x,\xi-\eta)b(x-y,\xi)dyd\eta$$

define symbols $a^* \in S^m$ and $a\#b \in S^{m+l}$ with the following asymptotic expansions:

$$a^* \sim \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \bar{a}$$

$$a\#b \sim \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!}\partial_\xi^\alpha a D_x^\alpha b$$

- Note that when $a(\xi), b(\xi)$ are polynomials in ξ (alternatively, $a(D), b(D)$ are linear partial differential operator), then $a* = \bar{a}$ and $a\#b = ab$

Lemma 4.7. *(Properties of * and $\#$)*

(i) $(a^*)^*$
(ii) $a\#1 = 1\#a = a$
(iii) $a\#(b\#c) = (a\#b)\#c$
(iv) $(a\#b)^* = b^*\#a^*$

Also, if $a, b \in S^\infty$ and ϕ and $\psi \in \mathcal{S}$, then

(v) $(a^*(x,D)\phi, \psi) = (\phi, a(x,D)\psi)$
(vi) $(a\#b(x,D)\phi, \psi) = (a(x,D)b(x,D)\phi, \psi)$

Consider a linear partial ifferential operator $a(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ with a_α smooth and complex-valued.

- Then $a(x,D)$ is said to be *locally solvable* at x_0 if there exists a neighborhood Υ of x_0 such that $a(x,D)u = f$ has a solution $u \in \mathcal{D}'(\Upsilon)$ for any $f \in C_0^\infty(\Upsilon)$
- $p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \in C^\infty(T^*(\mathbb{R}^n))$ is said to be the *principal symbol* of $a(x,D)$
- The *Poisson bracket* of two C^1 complex-valued functions on $T^*\mathbb{R}^n$ is given by

  $$\{p,q\}(x,\xi) = \langle \partial_\xi p(x,\xi), \partial_x q(x,\xi)\rangle - \langle \partial_x p(x,\xi), \partial_\xi q(x,\xi)\rangle$$

  In the case that p and q are the principal symbols of linear partial differential operators $a(x,D)$ and $b(x,D)$, respectively, $\{p,q\}(x,\xi)$ is the principal symbol (modulo a factor of i) of the commutator

  $$[a(x,D), b(x,D)] = (a\#b - b\#a)(x,D) = (ab - ba)(x,D)$$

  (see Lemma 4.13)
- $a(x,D)$ is said to be of *principal type* at x_0 if the ξ-gradient of its principal symbol at x_0 vanishes only for $\xi = 0$, that is, $\partial_\xi p(x_0,\xi) = 0$ if and only if $\xi = 0$.
- $a(x,D)$ is said to be *principally normal* at x_0 if there exists a function $q \in C^\infty(T^*\mathbb{R}^n \setminus \{0\})$ homogeneous of degree $m-1$ in ξ such that the principal symbol p satisfies

  $$\{\bar{p}, p\}(x,\xi) = 2i\text{Re }(\bar{q}(x,\xi)p(x,\xi))$$

  for $\xi \in \mathbb{R}^n \setminus \{0\}$ and x near x_0

4.2. **Solvability Theorem.**

Lemma 4.8. *(Garding's Inequality) Let* $a \in S^{2m}$ *and assume that for some* C_0 *and* $\epsilon > 0$ *and all* x, ξ *we have* $Re\ a(x,\xi) + C_0\lambda^{2m-1} \geq \epsilon\lambda^{2m}$. *Then for any* $N \geq 0$ *there exists a constant* C_N *such that for all* $\phi \in \mathcal{S}$

$$2Re\ (a(x,D)\phi,\phi) \geq \epsilon\|\phi\|_m^2 - C_N\|\phi\|_{m-N}^2$$

Proof. Let $b = \lambda^{-m}\#a\#\lambda^{-m} \in S^0$. Then since $b = \lambda^{-2m}a$ modulo S^{-1}, the hypotheses of the lemma imply that

$$\text{Re } b + (C_0 + C_1)\lambda^{-1} \geq \epsilon$$

for some $C_1 \in \mathbb{R}$, so that b satisfies the hypotheses of the lemma with $m = 0$. If the theorem is true in that case, then for $\phi \in \mathcal{S}$,

$$\begin{aligned} 2\text{Re } (a(x,D)\phi,\phi) &= 2\text{Re } (b(x,D)\lambda^m(D)\phi, \lambda^m(D)\phi) \\ &\geq \epsilon\|\lambda^m(D)\phi\|_0^2 - C_N\|\lambda^m(D)\phi\|_{-N}^2 \\ &= \epsilon\|\phi\|_m^2 - C_N\|\phi\|_{m-N}^2 \end{aligned}$$

and we are finished.

Hence it suffices to assume that $m = 0$, so that $a \in S^0$ with Re $a + C_0\lambda^{-1} \geq \epsilon$. Choose $F \in C^\infty(\mathbb{C})$ such that $F(z) = ((\epsilon/2) + z)^{1/2}$ for $z \in \mathbb{R}^+$. Since $2(\text{Re } a + C_0\lambda^{-1} - \epsilon) \in S^0$ is nonnegative, Lemma 4.1 implies that $b = (2\text{Re } a + 2C_0\lambda^{-1} - (3/2)\epsilon)^{1/2} = F(2(\text{Re } a + C_0\lambda^{-1} - \epsilon)) \in S^0$. Modulo S^{-1}, we have $b^*\#b = 2\text{Re } a - (3/2)\epsilon = a + a^* - (3/2)\epsilon$. In particular, for some $c \in S^{-1}$, we have

$$a + a^* = b^*\#b + \frac{3}{2}\epsilon + c$$

So if $\phi \in \mathcal{S}$,

$$\begin{aligned} 2\text{Re } (a(x,D)\phi,\phi) &= (a(x,D)\phi,\phi) + (\phi, a(x,D)\phi) \\ &= ((a+a^*)(x,D)\phi,\phi) \\ &= (b^*\#b(x,D)\phi,\phi) + (\frac{3}{2}\epsilon\phi,\phi) + (c(x,D)\phi,\phi) \\ &\geq \|b(x,D)\phi\|_0^2 + \frac{3}{2}\epsilon\|\phi\|_0^2 - \|c(x,D)\phi\|_{1/2}\|\phi\|_{-1/2} \\ &\geq \epsilon\|\phi\|_0^2 + (\frac{\epsilon}{2}\|\phi\|_0^2 - C_{1/2}\|\phi\|_{-1/2}^2) \end{aligned}$$

for some $C_{1/2} \in \mathbb{R}$ because $c \in S^{-1}$. So it suffices to prove

$$C_{1/2}\|\phi\|_{-1/2}^2 \leq \frac{\epsilon}{2}\|\phi\|_0^2 + C_N\|\phi\|_{-N}^2$$

where $C_N := \frac{\epsilon}{2}(\frac{2C_{1/2}}{\epsilon})^{2N}$. This can be seen as follows. When $C_{1/2}\lambda^{-1}(\xi) \geq \epsilon/2$, then $\lambda(\xi) \leq 2C_{1/2}/\epsilon$, so that

$$\begin{aligned} C_{1/2}\lambda^{-1}(\xi) &= C_{1/2}\lambda^{2N-1}(\xi)\lambda^{-2N}(\xi) \\ &\leq C_{1/2}(2C_{1/2}/\epsilon)^{2N-1}\lambda^{-2N}(\xi) \\ &= C_N\lambda^{-2N}(\xi) \\ &\leq \epsilon/2 + C_N\lambda^{-2N} \end{aligned}$$

The desired estimate is obtained after multiplication by $|\hat{\phi}|^2$ and integration. □

Lemma 4.9. *Let $\Upsilon_\delta = \{x \in \mathbb{R}^n : |x| < \delta\}$. Then for all $\delta > 0$, $m \in \mathbb{Z}_+$, we have*

$$\|\phi\|_m \leq 2\delta\|\phi\|_{m+1}$$

whenever $\phi \in C_0^\infty(\Upsilon_\delta)$. In addition, if Q and R are differential operators of orders m and $2m$, respectively, then there exists $C \in \mathbb{R}$ such that for all $\phi \in C_0^\infty(\Upsilon_\delta)$, we have

$$\|Q(ix_j\phi)\|_0 \leq C\delta\|\phi\|_m$$
$$|(ix_j, R\phi)| \leq C\delta\|\phi\|_m^2$$

Proof. Recall that $\|\phi\|_{s+1}^2 = \|\phi\|_s^2 + \sum_j \|D_j\phi\|_s^2$. Thus the first inequality follows immediately by induction once the case $m = 0$ is established. Since $\|D_1\phi\|_0 \leq \|\phi\|_1$, we have

$$\begin{aligned}
\|\phi\|_0^2 &= (\phi, \phi) \\
&= (D_1(ix_1\phi), \phi) - (ix_1(D_1\phi), \phi) \\
&= (ix_1\phi, D_1\phi) + (D_1\phi, ix_1\phi) \\
&\leq 2\|ix_1\phi\|_0\|D_1\phi\|_0 \\
&\leq 2\delta\|\phi\|_0\|\phi\|_1
\end{aligned}$$

For the second inequality, write $Q(ix_j\phi) = [Q, ix_j]\phi ix_j(Q\phi)$ so that

$$\begin{aligned}
\|Q(ix_j\phi)\|_0 &\leq \|[Q, ix_j]\phi\|_0 \\
&\leq C\|\phi\|_{m-1} + C\delta\|\phi\|_m
\end{aligned}$$

because $[Q, ix_j]$ has order $m - 1$, and the result follows from the first inequality.

For the third inequality, write $R = \sum_k Q_k Q'_k$ for some mth order operators Q_k and Q'_k. By the second inequality, we have

$$\begin{aligned}
|(ix_j\phi, R\phi)| &= |\sum_k (Q_k^*(ix_j\phi), Q'_k\phi)| \\
&\leq \sum_k \|Q_k^*(ix_j\phi)\|_0\|Q'_k\phi\|_0
\end{aligned}$$

□

Lemma 4.10. *Let $a(x, D)$ be a linear differential operator of order m. Then*

(i) If $a(x, D)$ is principal type at 0, there exists a $\delta_0 > 0$ and a C_0 such that for all $\delta < \delta_0$ and $\phi \in C_0^\infty(\Upsilon_\delta)$,

$$\|\phi\|_{m-1}^2 \leq C_0\delta(\|a(x, D)\phi\|_0^2 + \|a^*(x, D)\phi\|_0^2 + \|\phi\|_{m-1}^2)$$

(ii) If $a(x, D)$ is principally normal at 0, there exists a $\delta > 0$ and a $C \in \mathbb{R}$ such that for all $\phi \in C_0^\infty(\Upsilon_\delta)$,

$$\|a(x, D)\phi\|_0^2 \leq C(\|a^*(x, D)\phi\|_0^2 + \|\phi\|_{m-1}^2)$$

(iii) If $a(x, D)$ is both principally normal and of principal type at 0, there exists a $\delta > 0$ such that for all $\phi \in C_0^\infty(\Upsilon_\delta)$,

$$\|\phi\|_{m-1} \leq \|a^*(x, D)\phi\|_0$$

Proof.

(i) Let $A = a(x, D)$, $Q_j = [A, ix_j] = (\partial_{\xi_j} a)(x, D)$, $B = \sum_{j=1}^n Q_j^* Q_j = b(x, D)$. By Lemma 4.6, $b = \sum_{j=1}^n |\partial_{\xi_j} p|^2$ modulo S^{2m-3}. As A is of principal type, homogeneity gives $\sum_{j=1}^n |\partial_{\xi_j} p(x, \xi)|^2 \geq 2\epsilon|\xi|^{2m-2}$ for some $\epsilon > 0$ and all $x \in \Upsilon_{2\delta_0}$. Hence the symbol $b + \epsilon\lambda^{2m-2}\#(1-\psi)$ satisfies the hypothesis of Garding's inequality, provided $\phi \in C_0^\infty(\Upsilon_{2\delta_0})$ and $\psi \leq 1$. If also $\psi = 1$ in Υ_{δ_0} and $\delta < \delta_0$ is such that $(1-\psi)\phi = 0$ for all $\phi \in C_0^\infty(\Upsilon_\delta)$, then $(b(x, D) + \epsilon\lambda^{2m-2}(D)(1-\psi))\phi = B\phi$. Thus we have

$$2\sum_{j=1}^n Q_j\phi_0^2 = 2Re(b\phi, \phi) \geq \epsilon\|\phi\|_{m-1}^2 - C\|\phi\|_{m-2}^2$$

for some $C \in \mathbb{R}$. However, for each operator Q_j we have

$$\begin{aligned}\|Q_j\phi\|_0^2 &= (A(ix_j\phi) - ix_j(A\phi), Q_j\phi)\\ &= (ix_j\phi, A^*Q_j\phi) - (ix_j(A\phi), Q_j\phi)\\ &= (ix_j\phi, [A*, Q_j]\phi) + (Q_j^*(ix_j\phi), A^*\phi) - (ix_j(A\phi), Q_j\phi)\end{aligned}$$

If $\phi \in C_0^\infty$, Lemma 4.9 gives that

$$\begin{aligned}\|Q_j\phi\|_0^2 &\leq C_{j,1}\delta\|\phi\|_{m-1}^2 + C_{j,2}\delta\|\phi\|_{m-1}\|A^*\phi\|_0 + C_{j,3}\delta\|A\phi\|_0\|\phi\|_{m-1}\\ &\leq C_j\delta(\|A\phi\|_0^2 + \|A^*\phi\|_0^2 + \|\phi\|_{m-1}^2)\end{aligned}$$

We also have $\|\phi\|_{m-2}^2 \leq 4\delta^2\|\phi\|_{m-1}^2$ by Lemma 4.9. Hence

$$\begin{aligned}\|\phi\|_{m-1}^2 &\leq \frac{2}{\epsilon}\sum_{j=1}^n \|Q_j\phi\|_0^2 + \frac{C}{\epsilon}\|\phi\|_{m-2}^2\\ &\leq C_0\delta(\|A\phi\|_0^2 + \|A^*\phi\|_0^2 + \|\phi\|_{m-1}^2)\end{aligned}$$

as desired.

(ii) Modify the function q near $\xi = 0$ so that $q \in C^\infty$ everywhere while $\{\bar{p}, p\} = 2i\text{Re }(\bar{q}p)$ holds only for $|\xi| \geq 1$ and x in some $\Upsilon_{2\delta}$. Then for $\phi \in C_0^\infty(\Upsilon_{2\delta})$ we define

$$\begin{aligned}b &= \phi a \in S^m\\ c &= \phi q + i\{a, \phi\} \in S^{m-1}\\ r &= b^*\#b - b\#b^* - b\#c^* - c\#b^* \in S^{2m}\end{aligned}$$

Indeed, we see that $r \in S^{2m-2}$ via Lemma 4.6. More precisely, modulo S^{2m-2}, we have $b^* = \bar{b} - i\bar{b}_{\langle x,\xi\rangle}$, so that

$$\begin{aligned}r &= (\bar{b} - i\bar{b}_{\langle x,\xi\rangle})b - i\langle\bar{b}_\xi, b_x\rangle - b(\bar{b} - i\bar{b}_{\langle x,\xi\rangle}) + i\langle b_\xi, \bar{b}_x\rangle - b\bar{c} - c\bar{b}\\ &= -\{\bar{b}, b\} - 2\text{Re }(\bar{c}b)\\ &= -i(\{\bar{b}, b\} - 2i\text{Re }(\bar{c}b))\\ &= -i\phi^2(\{\bar{a}, a\} - 2i\text{Re }(\bar{q}a))\\ &= -i\phi^2(\{\bar{p}, p\} - 2i\text{Re }(\bar{q}, p))\end{aligned}$$

which is zero when $|\xi| \geq 1$. Thus, if ψ is chosen such that $\psi = 1$ in Υ_δ, then $B\phi = A\phi$ and $B^*\phi = A^*\phi$ for all $\phi \in C_0^\infty(\Upsilon_\delta)$ because A has the local property. Hence we have

$$\begin{aligned}\|A\phi\|_0^2 &= (B^*B\phi,\phi)\\ &= (R\phi,\phi)+(BB^*\phi,\phi)+(BQ^*\phi,\phi)+(QB^*\phi,\phi)\\ &= (R\phi,\phi)+\|A^*\phi\|_0^2+2\mathrm{Re}\,(Q^*\phi,A^*\phi)\\ &\le \|R\phi\|_{1-m}\|\phi\|_{m-1}+2\|A^*\phi\|_0^2+\|Q^*\phi\|_0^2\\ &\le 2\|A^*\phi\|_0^2+C\|\phi\|_{m-1}^2\end{aligned}$$

since $R\in\Psi^{2m-2}$ and $Q^*\in\Psi^{m-1}$.

(iii) Finally, if both hypotheses are valid, (i) and (ii) imply that for small $\delta>0$ and for $\phi\in C_0^\infty(\Upsilon_\delta)$,

$$\|\phi\|_{m-1}^2\le C_1\delta(\|a^*(x,D)\phi\|_0^2+\|\phi\|_{m-1}^2)$$

for some C_1. If $\delta<1/2C_1$, then

$$\begin{aligned}\|\phi\|_{m-1}^2 &= 2\|\phi\|_{m-1}^2-\|\phi\|_{m-1}^2\\ &\le 2C_1\delta(\|a^*(x,D)\phi\|_0^2+\|\phi\|_{m-1}^2)-\|\phi\|_{m-1}^2\\ &\le \|a^*(x,D)\phi\|_0^2\end{aligned}$$

□

Theorem 4.11. *Let $a(x,D)$ be a principally normal operator of order m and of principal type at x_0. Then there exists a neighborhood Υ of x_0 such that the equation $a(x,D)u=f$ has a solution $u\in L^2(\Upsilon)$ for any $f\in H^{1-m}$.*

Proof. Using translation we can assume without loss of generality that $x_0=0$, and take $\delta>0$ as in Lemma 4.10(iii). Then $a^*(x,D)$ is injective on $C_0^\infty(\Upsilon_\delta)$, and so its inverse $(A^*)^{-1}$ is well defined on

$$\mathbb{E}=\{\psi\in C_0^\infty(\Upsilon_\delta):\exists\phi\in C_0^\infty(\Upsilon_\delta)\text{ with }\psi=a^*(x,D)\psi\}$$

For each $f\in H^{1-m}$ define the semilinear form $U_f(\psi)=(f,(A^*)^{-1}\psi)$ on $\mathbb{E}$. Using Lemma 4.10.iii on $\phi=(A^*)^{-1}\psi$, we have

$$\begin{aligned}|U(\psi)| &= |(f,\phi)|\\ &\le \|f\|_{1-m}\|\phi\|_{m-1}\\ &\le \|f\|_{1-m}\|a^*(x,D)\phi\|_0\\ &= \|f\|_{1-m}\|\psi\|_0\end{aligned}$$

Hence U is continuous in the L^2-norm. By the Hahn-Banach theorem, U extends continuously to $L^2(\Upsilon_\delta)$, and by the Riesz Representation Theorem there exists $u\in L^2(\Upsilon_\delta)$ such that $(u,\phi)=U(\phi)$ for $\phi\in\mathbb{E}$. In particular, $(u,a^*(x,D)\phi)=(f,\phi)$ for all $\phi\in C_0^\infty(\Upsilon_\delta)$, so that $a(x,D)u=f$ in Υ_δ. □

4.3. **Converse to Solvability Theorem.** Let $\Upsilon\subset\mathbb{R}^n$, and consider the differential operator

$$a(x,D)=\sum_{|\alpha|\le m}a_\alpha(x)D^\alpha$$

of order m with coefficients in $C^\infty(\Upsilon)$, and let p be its principal symbol (note that here D is a differential operator on x, not ξ). Also, let $\bar{p}$ be the corresponding symbol with conjugate coefficients $\bar{a}_\alpha$. That is,

$$p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$$

$$\bar{p}(x,\xi) = \sum_{|\alpha|=m} \bar{a}_\alpha(x)\xi^\alpha$$

We also define

$$C_{2m-1}(x,\xi) = \sum_{j=1}^{n} i(\partial_{\xi_j} p(x,\xi)\bar{\partial}_{x_j} p(x,\xi) - \partial_{x_j} p(x,\xi)\partial_{\xi_j}\bar{p}(x,\xi)) \tag{21}$$
$$= \{p, \bar{p}\}$$

Then C_{2m-1} is a polynomial in ξ of degree $2m-1$ with real coefficients, and is the principal symbol of the commutator $[a, \bar{a}]$.

Theorem 4.12. (Due to Hormander, [7]) *Suppose $a(x,D)u = f$ has a solution $u \in \mathcal{D}'(\Upsilon)$ for every $f \in C_0^\infty(\Upsilon)$. If $x \in \Upsilon, \xi \in \mathbb{R}^n$ are such that $p(x,\xi) = 0$, then $C_{2m-1}(x,\xi) = 0$ also.*

The proof will require some preliminary results.

Lemma 4.13. *Let*

$$C(x,D) = \bar{a}(x,D)a(x,D) - a(x,D)\bar{a}(x,D) = [a,\bar{a}]$$

Then $C(x,D)$ is of order at most $2m-1$ and $C_{2m-1}(x,D)$ is the sum of the terms in $C(x,D)$ of order $2m-1$. That is,

$$C(x,D) = C_{2m-1}(x,D) + \text{ terms of order } \leq 2m-1$$

Proof. Recall Leibniz's rule, Proposition A.5. Namely, given $u \in \mathcal{D}'$, $b \in C^\infty$, and $a(D)$ a polynomial in the variables $\xi_1, \cdots, \xi_n$, with ξ_j replaced by D_j, then

$$a(D)(bu) = \sum_\alpha (D^\alpha b)((\partial_\xi^\alpha a)(D)u)/\alpha!$$

Thus we obtain

$$\bar{a}(x,D)a(x,D) = \sum_\alpha \sum_\beta (D^\alpha b^\beta(x)/\alpha!)\bar{a}^{(\alpha)}(x,D)D^\beta$$

and a similar formula holds for $a(x,D)\bar{a}(x,D)$. Thus

$$C(x,\xi) = \sum_{\alpha\neq 0} (\partial_\xi^\alpha \bar{a}(x,\xi)D^\alpha a(x,\xi) - \partial_\xi^\alpha a(x,\xi)D^\alpha \bar{a}(x,\xi))/\alpha!$$

where D acts on x. Here the terms where $\alpha = 0$ cancel and thus can be omitted from the sum. Moreover, the $\alpha = 0$ terms are the only terms of order $2m$. Thus $C(x,\xi)$ is of order at most $2m-1$ and the terms of order $2m-1$ are given by C_{2m-1}. ☐

Lemma 4.14. *Assume the hypotheses of Theorem 4.12, and let $\upsilon \subset\subset \Upsilon$ be an open set. Then there exist constants C,k,N such that*

$$|\int fv dx| \leq C \sum_{|\alpha|\leq k} \sup_{x\in\upsilon} |D^\alpha f| \sum_{|\beta|\leq N} \sup_{x\in\upsilon} |D^\beta\ {}^t av| \tag{22}$$

when $f, v \in C_0^\infty(\upsilon)$.

Proof. We consider $\int fvdx$ as a bilinear form for $f \in C_0^\infty(\bar{\upsilon})$ and $v \in C_0^\infty(\upsilon)$. Here C_0^∞ is the Frechet space with the topology from the semi-norms $\sup_{x\in\upsilon} |D^\alpha f(x)|$ and $C_0^\infty(\upsilon)$ with the (metrizable) topology from the semi-norms $\sup_{x\in\upsilon} |D^\beta\ {}^t av|$. This bilinear form is clearly continuous in f for fixed v. When f is fixed, we can by hypothesis take $u \in \mathcal{D}'(\Upsilon)$ such that $P(x, D)u = f$. Thus

$$\int fvdx = \int (au)(v) = \int u({}^t av)$$

so that the form is continuous in v for fixed f. A bilinear form on a product of a Frechet space and a metrizable space is continuous if provided it is seperately continuous, so we are finished. □

Lemma 4.15. *Given* $(a_1, \cdots, a_n), (f_1, \cdots, f_n) \in \mathbb{C}^n$, *where some* $a_j \neq 0$, *there exists a symmetric matrix* $A = (\alpha_{jk})$ *with positive definite imaginary part satisfying*

$$Aa = \sum_{j=1}^{n} \alpha_{jk} a_j = f_k, 1 \leq k \leq n \tag{23}$$

if and only if

$$Im \sum_{k=1}^{n} f_k \bar{a}_k > 0 \tag{24}$$

Proof. First, we show that condition (23) implies (24). If $b_j = \text{Re } a_j$ and $c_j = \text{Im } a_j$ then the symmetry of α_{kj} and condition (23) give that

$$(f, a) = \sum_{k=1}^{n} f_k \bar{a}_k = \sum_{j,k=1}^{n} \alpha_{kj} a_j \bar{a}_k = \sum_{j,k=1}^{n} \alpha_{kj} b_j b_k + \sum_{j,k=1}^{n} \alpha_{kj} c_j c_k$$

The real vectors $(b_1, \cdots, b_n)$ and $(c_1, \cdots, c_n)$ do not both vanish and (Im α_{kj}) is positive definite, thus (24) is established.

Second, we show show that (24) implies (23). There are two cases to consider.

(1) Assume $ca \in \mathbb{R}^n$ for some constant $c \in \mathbb{C}$. Replacing a and f with ca and cf, respectively, we may assume that $a \in \mathbb{R}^n$. Writing $\alpha = \beta + i\gamma$ and $f = g + ih$, then (23) can be rewritten as

$$\beta a = g, \gamma a = h$$

Certainly we can find a real symmetric matrix β with $\beta a = g$. To see this, we will use a simple induction on n.

- *Base case:* Let $n = 1$. Then since a is nonzero, a_1 is nonzero, and so if we take $\beta = g_1/a_1$ we are finished.
- *Induction step:* Assume that the result holds for all $n \leq k$, and now take $n = k + 1$. Since a is nonzero, one of $a_1, \cdots, a_n$ is nonzero. If a_1 is the only nonzero component of a, then the result is trivial. So assume that one of $a_2, \cdots, a_n$ is nonzero. Thus we must find a symmetric matrix $\beta = (\beta_{ij})$ such that the following system of equations is

satisfied:

$$\begin{aligned}\beta_{11}a_1+\beta_{12}a_2+\cdots+\beta_{1n}a_n&=g_1\\ \beta_{12}a_1+\beta_{22}a_2+\cdots+\beta_{2n}a_n&=g_2\\ &\cdots\\ \beta_{1n}a_1+\beta_{2n}a_2+\cdots+\beta_{nn}a_n&=g_n\end{aligned}$$

Clearly we can choose $\beta_{11},\cdots,\beta_{1n}$ so that the first equation is satisfied. Thus we are reduced to solving the system

$$\begin{aligned}\beta_{22}a_2+\cdots+\beta_{2n}a_n&=g_2-\beta_{12}a_1\\ \beta_{23}a_2+\cdots+\beta_{3n}a_n&=g_3-\beta_{13}a_1\\ &\cdots\\ \beta_{2n}a_2+\cdots+\beta_{nn}a_n&=g_n-\beta_{1n}a_1\end{aligned}$$

where one of $a_2,\cdots,a_n$ is nonzero, and $\beta_{12},\cdots,\beta_{1n}$ have been fixed (β is symmetric). This is the problem in the case $n=k$, and the induction is complete.

Next, let $h_1=h-a\frac{(h,a)}{2(a,a)}$. Then $(h_1,a)=(h,a)/2>0$. Thus if we define γ by $\gamma x=\frac{(h,a)}{(2a,a)}x+\frac{(x,h_1)}{(a,h_1)}h_1$, γ will be positive definite. From the definition of h_1 we see that $\gamma a=h$.

(2) Assume $ca\notin\mathbb{R}^n$ for any $c\in\mathbb{C}$. It suffices to show that

$$\alpha=i\frac{\text{Im }(f,a)}{(a,a)}I+\beta$$

satisfies (23) for some real symmetric β. So we must have

$$\beta a=f-ai\text{Im }\frac{(f,a)}{(a,a)}=:f_1 \tag{25}$$

with

$$\text{Im }(f_1,a)=0 \tag{26}$$

So it remains to find a β. To prove that such a β exists, notice that $\{z\in\mathbb{C}^n:\exists$ symmetric γ such that $z=\gamma a\}$ is a linear subspace with respect to real scalars. The equation of a plane containing this set can be written as $\text{Im }(z,g)=0$ for some $g\in\mathbb{C}^n$. Let β be defined by $\beta x=\xi(x,\xi)$. Then β is real and symmetric for every $\xi\in\mathbb{R}^n$, and $\beta a=\xi(a,\xi)$. Thus

$$\text{Im }(\xi,g)(a,\xi)=0$$

By assumption, a is not proportional to any real vector. Thus g must be a real multiple of a, and $\text{Im }(z,g)=0$ follows from the requirement that $\text{Im }(z,a)=0$. Thus by (26) there is a real symmetric matrix β satisfying (25).

□

If we can show that when the conclusion of Theorem 4.12 is not satisfied, the conclusion of Lemma 4.14 is not valid for any C,k,n, we will have proved Theorem 4.12. Assume without loss of generality that $0\in\Upsilon$ and the conclusion of Theorem

4.12 is not valid when $x = 0$. Since $C_{2m-1}(0, \xi)$ is real valued and odd for $\xi \in \mathbb{R}$, we can find a ξ such that

$$\xi \in \mathbb{R}^n/\{0\},\ p(0, \xi) = 0,\ C_{2m-1}(0, \xi) < 0 \tag{27}$$

Lemma 4.16. *Assume condition (27), and let $q \in \mathbb{Z}^+$. Then there exists $w \in C^\infty(\Upsilon)$, depending on q, such that*

$$p(x, grad\ w) = O(|x|^q),\ \ as\ x \to 0 \tag{28}$$

$$w(x) = \langle x, \xi\rangle + \frac{1}{2}\sum_{j,k=1}^{n} \alpha_{jk} x_j x_k + O(|x|^3),\ \ as\ x \to 0 \tag{29}$$

where the matrix α_{jk} is symmetric and has a positive definite imaginary part.

Proof. (28) holds when $q = 1$ if $w(x) = \langle x, \xi\rangle$ since $w(x)$ then satisfies (29), grad $w(x) = (\xi_1, \cdots, \xi_n)$, and $p(0, \xi) = 0$ so that (28) is satisfied as well. In order for (28) to hold when $q = 2$, we have to choose α_{jk} such that the first order derivatives of $p(x, \text{grad}\ w)$ are zero at 0, i.e.

$$\partial_{x_j} p(0, \xi) + \sum_{k=1}^{m} \partial_{\xi_k} p(0, \xi)\alpha_{jk} = 0, 1 \le j \le n \tag{30}$$

By Lemma 4.15, equation (21), and equation (27), there exists a symmetric matrix α_{jk} with positive definite imaginary part which satisfies (30). Thus we can prove (28) for an arbitrary q as follows. First, assume the coefficients of p are analytic, as (28) and (29) do not change if the coefficients of p are replaced by their Taylor expansions of order q. Since $C_{2m-1} < 0$ we have $\partial_{\xi_j} p(0, \xi) \neq 0$ for some j, say $j = n$. By Theorem 1.8.2 of [7] and the ensuing discussion we can thus find a solution W of $p(x, \text{grad}\ W) = 0$ near 0, so that grad $W(0) = \xi$ and $W(x) = \langle x, \xi\rangle + \frac{1}{2}\sum_{j,k=1}^{n} \alpha_{jk} x_j x_k$ when $x_n = 0$. Since

$$\partial_{x_j} p + \sum_{k=1}^{n} \partial_{\xi_k} p(0, \xi)\partial_{x_j}\partial_{x_k} W(0) = 0, 1 \le j \le n \tag{31}$$

and $\partial_{x_j}\partial_{x_k} W(0) = \alpha_{jk}$ if $j, k < n$, (30) and (31) with $j < n$ give that $\partial_{x_j}\partial_{x_n} W(0) = \alpha_{jn}$ if $j < n$. Applying the same formulas with $j = n$ gives that $\partial^2_{x_n} W(0) = \alpha_{nn}$. Hence W satisfies (29). If $\phi \in C_0^\infty$ is 1 in a neighborhood of the origin and supported in the set where W is defined, then $w = \phi W$ satisfies the requirements of the lemma. □

Now we are prepared to prove the main theorem.

Proof. (of Theorem 4.12) As mentioned above, we argue by contradiction. In particular, assume that the hypotheses of Theorem 4.12 are true but the conclusion is false. We will show this implies that for all C, k, N, the conclusion of Lemma 4.14 does not hold when υ is a neighborhood of zero, a contradiction. Choose w via Lemma 4.16, with

$$q = 2r, r = n + m + k + N + 1 \tag{32}$$

Let $\phi_0, \cdots, \phi_{r-1} \in C_0^\infty(\upsilon)$ and $F \in C_0^\infty(\mathbb{R}^n)$ be functions (yet to be determined), and set

$$v_\tau = \tau^{n+1+k} e^{i\tau w} \sum_{\nu=0}^{r-1} \phi_\nu \tau^{-\nu}$$
$$f_\tau(x) = \tau^{-k} F(\tau x)$$

τ is a parameter which will tend to ∞. The idea is to choose the ϕ_ν and F so that the right side of equation (22) is bounded independent of τ while the left side of the equation can be made arbitrarily large.

When τ is large, $f_\tau \in C_0^\infty(\upsilon)$ (as υ is a neighborhood of zero) and $v_\tau \in C_0^\infty(\upsilon)$ for each τ. Through change of variables, we see that

$$\tau^{-1} \int f_\tau v_\tau dx = \int F(x) e^{i\tau w(x/t)} (\sum_{\nu=0}^{r-1} \phi_\nu (x/\tau) \tau^{-\nu}) dx$$

Since supp F is compact and the right-side integrand is uniformly convergent on supp F to the limit $F(x)e^{i\langle x,\xi\rangle}\phi_0(0)$, the right side integral has limit $\hat{F}(-\xi)\phi_0(0)$ when $\tau \to \infty$. If F and ϕ_0 chosen so that $\hat{F}(-\xi) \neq 0$ and $\phi_0(0) = 1$, we get

$$\int f_\tau v_\tau dx \to \infty, \tau \to \infty$$

We also have that when $|\alpha| \leq k$ and $\tau \geq 1$,

$$\sup_{\mathbb{R}^n} |D^\alpha f_\tau| \leq \sup_{\mathbb{R}^n} |D^\alpha F|$$

Thus to prove that the conclusion of Lemma 4.14 is false it remains to show that we can choose $\phi_0, \cdots, \phi_{r-1}$ and C such that

$$\sup_{x \in \upsilon} |D^\alpha\ {}^t P v_\tau| \leq C, \tau \geq 1, |\alpha| \leq N \tag{33}$$

Now, when $\psi \in C^\infty$ we have by Leibniz's rule, Proposition A.5, that

$${}^t P(\psi e^{i\tau w}) = \sum_{j=0}^{m} c_j \tau^j e^{i\tau w} \tag{34}$$

where the $c_j \in C^\infty$ are independent of τ.

Next, note that the principal part $q(x, D)$ of ${}^t a(x, D)$ is $q(x, D) = p(x, -D)$. This can be seen as follows. First note that if $R(x, D)$ is a differential operator of order k, then repeated integration by parts gives that ${}^t R(x, D)$ is also of order k. Hence it suffices to show that ${}^t p(x, D) = p(x, -D)$. This also follows by repeated integration by parts, together when an induction on the order of p.

Thus by Leibniz's formula, Proposition A.5, we have

$$c_m = A\psi, c_{m-1} = \sum_{j=1}^{n} A_j D_j \psi + B\psi \tag{35}$$

where $A = p(x, -\text{grad}\ w)$ and $A_j = -\partial_{\xi_j} p(x, -\text{grad}\ w)$. The specific choice of $B \in C^\infty$ is not of concern, however, it *is* independent of ψ. By equations (28) and (32), we have that for x near 0,

$$A(x) = O(|x|^{2r})$$

Also, equation (27) says that for some $j \neq 0$, $A_j(0) \neq 0$. If we take $\psi = \phi_\nu$ and notice that $n+1+k+m = r-N$, equations (34), (35) show that

$$ {}^t a v_\tau = \tau^{r-N} e^{i\tau w} \sum_{\mu=0}^{m+r-1} \alpha_\mu \tau^{-\mu} \tag{36} $$

where

$$ a_0 = A\phi_0, a_1 = A\phi_1 + \sum_{j=1}^{n} A_j D_j \phi_0 + B\phi_0 $$

The general form of the coefficients a_μ is given by

$$ a_\mu = A\phi_\mu + \sum_{j=1}^{n} A_j D_j \phi_{\mu-1} + B\phi_{\mu-1} + L_\mu \tag{37} $$

provided ϕ_ν is interpreted as 0 when $\nu \geq r$. Here L_μ is a linear combination of functions ϕ_ν with $\nu < \mu - 1$ and their derivatives.

Next we choose the functions $\phi_\nu \in C_0^\infty(\upsilon)$. In particular, we show that the ϕ_ν can be choosen so that $\phi_0(0) = 1$ and

$$ a_\mu(x) = O(|x|^{2(r-\mu)}), \mu \leq r, x \to 0 \tag{38} $$

When $\mu = 0$, the above equation (38) is a consequence of (4.3). Equation (4.3) also gives that the first term in (37) does not affect (38). So we must find $\phi_{\mu-1}$ such that

$$ \sum_{j=1}^{n} A_j D_j \phi_{\mu-1} + B\phi_{\mu-1} + L_\mu = O(|x|^{2(r-\mu)}) \tag{39} $$

Suppose all ϕ_ν have been chosen when $\nu < \mu - 1$ and $1 \leq \mu \leq r$. To choose $\phi_{\mu-1}$ we can assume A_j,B, and L_μ are analytic, as (39) still holds if the infinitely differentiable functions are replaced with Taylor expansions of order $2r$ about 0. By the Cauchy-Kowalevsky Theorem, we can find a solution $\Phi_{\mu-1}$ to

$$ \sum_{j=1}^{n} A_j D_j \Phi_{\mu-1} + B\Phi_{\mu-1} + L_\mu = 0 $$

in a neighborhood V of 0. Indeed, we can even choose the values of $\Phi_{\nu-1}$ on a noncharacteristic plane through 0. Note that such planes exist as $A_j(0) \neq 0$ for some j. Let $\eta \in C_0^\infty(\upsilon \cap V)$ be 1 near 0. Then $\phi_{\mu-1} := \Phi_{\mu-1}\eta \in C_0^\infty$ and satisfies equation (39). Note that when $\mu = 1$ we can easily satisfy the requirement $\phi_0(0) = 1$.

We will have satisfied (33) once we use the following lemma with equations (36) and (38). $\square$

Lemma 4.17. *If υ is a sufficiently small neighborhood of 0, $0 \leq s \in \mathbb{R}$ then*

$$ \sup_{x \in \upsilon} |D^\alpha(\psi(x) e^{i\tau w(x)})| = O(\tau^{|\alpha|-s}), \tau \to \infty $$

for every $\psi \in C_0^\infty(\upsilon)$ such that

$$ \psi(x) = O(|x|^{2s}), x \to 0 $$

Proof. By construction, the Taylor expansion of Im w at 0 begins with a positive definite quadratic form. Thus when υ is small, we have

$$\text{Im } w(x) \geq a|x|^2, x \in \upsilon$$

for some positive number a.

By Leibniz's formula, Proposition A.5, it suffices to show

$$\sup_{x \in \upsilon} |e^{i\tau w(x)} D^\beta \psi(x)| = O(\tau^{|\beta|-s})$$

as $\tau \to \infty$. Since Im $w(x) \geq 0$ in υ, this holds when $|\beta| \geq s$. When $\beta < s$, we see that

$$D^\beta \phi(x) = O(|x|^{2s-|\beta|}) = O(|x|^{2(s-|\beta|)})$$

for $x \in \upsilon$. Thus we have

$$\tau^{s-|\beta|}|e^{i\tau w} D^\beta \psi| \leq C(\tau|x|^2)^{s-|\beta|} e^{-a\tau|x|^2}$$

Here the right hand side is bounded in $\tau|x|^2$, and so we are done. □

In particular, we have

Corollary 4.18. *Let $a(x, D)$ be a linear differential operator with principal symbol p such that the real and imaginary parts of the ξ-gradient of p are linearly independent at (x_0, ξ) for all solutions $\xi \neq 0$ of $p(x_0, \xi) = 0$. Then $a(x, D)$ is of principal type at x_0, and the following are equivalent:*

(*i*) *$a(x, D)$ is principally normal at x_0*
(*ii*) *$a(x, D)$ is locally solvable at x_0*
(*iii*) *$a(x, D)$ satisfies $\{\bar{p}, p\} = 0$ on $p = 0$ in a neighborhood of x_0*

Proof. Since p is homogeneous of order m in ξ, Euler's Theorem gives $p(x_0, \xi) = (1/m)\langle \partial_\xi p(x_0, \xi), \xi \rangle$. Thus to show that a is of principal type at x_0, it suffices to see that $\partial_\xi p(x_0, \xi) \neq 0$ when $p(x_0, \xi) = 0$ and $\xi \neq 0$. This is guaranteed by hypothesis.

The implication (i)⇒(ii) follows from Theorem 4.11 and the implication (ii)⇒(iii) follows from Theorem 4.12. The implication (iii)⇒(i) is as follows. To show that a is principally normal at x_0, it suffices to check that a satisfies the defnition of principally normal near the zeroes of p. For if $p(x_0, \xi_0) \neq 0$, we can take $q = \frac{\{\bar{p}, p\}}{2i\bar{p}}$ and we have $\{\bar{p}, p\} = 2i\text{Re }(\bar{q}p)$. Hence if we can write $\{\bar{p}, p\} = 2i\text{Re }(\bar{q}p)$ near any zero of p, the compact set $K = \{(x, \xi) \in \mathbb{R}^{2n} : x = x_0, |\xi| = 1\}$ can be covered by finitely many open sets where $\{\bar{p}, p\} = 2i\text{Re }(\bar{q}_j p)$. Employing a partition of unity, we find a function q_0 such that $\{\bar{p}, p\} = 2i\text{Re }(\bar{q}_0, p\}$ in a neighborhood of K. Setting $q(x, \xi) = |\xi|^{m-1} q_0(x, \xi/|\xi|))$ we have $\{\bar{p}, p\} = 2i\text{Re }(\bar{q}p)$ for $\xi \in \mathbb{R}^n$ and x near x_0, by homogeneity.

Now, for (x, ξ') near (x_0, ξ), the hypotheses of the corollary give that Re p and Im p can be taken as local coordinates in $\mathbb{R}^{2n}$. By Taylor's formula,

$$\frac{1}{2i}\{\bar{p}, p\} = \frac{1}{2i}\{\bar{p}, p\}|_{p=0} + q_1 \text{Re } p + q_2 \text{Im } p$$

for some $q_1, q_2 \in C^\infty(\mathbb{R}^{2n})$. Taking $q = q_1 + q_2$, condition (iii) gives $\{\bar{p}, p\} = 2i\text{Re }(\bar{q}p)$. □

5. Other Results

5.1. **More General Linear PDE.** Charles Fefferman and Richard Beals proved the following general result in [1]. The following discussion is based on their paper.

Let a be a linear partial differential operator of order m, defined on a neighborhood Υ of $x_0 \in \mathbb{R}^{n+1}$. Assume that a is of principal type. Define the *bicharacteristic curves* of Re p to be the of the Hamilton Jacobi equations,

$$\frac{dx}{ds} = \partial_\xi(\text{Re } p)$$
$$\frac{d\xi}{ds} = -\partial_x(\text{Re } p)$$

on $\Upsilon \times (\mathbb{R}^{n+1}/\{0\})$. Re p is constant on bicharacteristics. We define the *null bicharacteristics* to be the bicharacteristics on which Re p is zero. An important condition used in the theorem is condition $(\mathcal{P})$ given by Nirenberg and Treves, namely, that Im p does not change sign along the null bicharacteristics of Re p.

Theorem 5.1. *Let a be a linear partial differential operator or order m with smooth coefficients defined on Υ. If a is of principal type and satisfies condition $(\mathcal{P})$, then for each real $s \geq 0$ there is a neighborhood Υ_s of x_0 such that $au = f$ has a solution $u \in H^{s+m-1}(\Upsilon_s)$ for every $f \in H^s(\Upsilon_s)$.*

5.2. **Nirenberg-Treves Conjecture.** In 1970, Nirenberg and Treves, [11], [12], made the following conjecture similar to Theorem 5.1:

Theorem 5.2. (Nirenberg-Treves Conjecture) *Let a be a pseudo-differential operator of principal type, and $x_0 \in \mathbb{R}^n$ be fixed. Also, let p denote the principal symbol of a (one can make sense of principal symbols for pseudo-differential operators in addition to linear partial differential operators). Then the following two statements are equivalent:*

- (i) *For any $f \in C^\infty$, there is some neighborhood of V_f of x_0 and some distribution $u \in \mathcal{D}'(V_f)$ such that $au = f$*
- (ii) (Condition (Ψ)) *If Im p is negative at a point on any null bicharacteristic Γ of Re p, then Im p remains nonpositive along Γ.*

(Note that the pseudo-differential operators in this Theorem are slightly different than the ones used in this paper) In their papers, Nirenberg and Treves proved that condition (Ψ) was necessary for local solvability.

Recently, Nils Dencker [2] has proven that condition (Ψ) is also sufficient for local solvability, thus resolving the Nirenberg-Treves conjecture.

Appendix A. Background Results

Proof. (of Lemma 4.1) Write $a = b + ic$, where b and c are real valued. Since $a \in S^0 \subset C^0 \cap L^\infty$, we have $F(a)$ is such that

$$|F^{(n)}(a)| \leq C_n \tag{40}$$

for all $n \in \mathbb{Z}_+$ (since a is bounded). To show that $F(a) \in S^0$, we must show that

$$|(\partial_x^\alpha \partial_\xi^\beta) F(a(x,\xi))| \leq C_{\alpha\beta}(1 + |\xi|^2)^{(m-|\beta|)/2} \tag{41}$$

for all multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$.

For notational convenience, let T^m, $m \in \mathbb{R}$, denote the space of all $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ functions $b(x,\xi)$ such that

$$|b(x,\xi)| \leq C(1+|\xi|^2)^{m/2}$$

for some constant C.

By definition of S^m, to prove the result it suffices to show that $\partial_x^\alpha \partial_\xi^\beta F(a(x,\xi)) \in T^{-|\beta|}$ for all $\alpha, \beta \in \mathbb{Z}_+^n$. First note that each S^m (respectively T_m) is a vector space, so a linear combination of terms in S^m (respectively T_m) is again in S^m (respectively T_m). Thus it suffices to show that $\partial_x^\alpha \partial_\xi^\beta F(a)$ is a linear combination of terms in $T^{-|\beta|}$. To do so, we will use induction.

Claim: Let $n = |\alpha| + |\beta|$. Then $\partial_x^\alpha \partial_\xi^\beta F(a)$ is a linear combination of terms of the form

$$F^{(k)}(a(x,\xi)) \prod_{i=1}^{k} (\partial_x^{\alpha_i} \partial_\xi^{\beta_i} a)(x,\xi) \tag{42}$$

for some $k \geq 0$ and multi-indices $\alpha_1, \beta_1, \cdots, \alpha_{n-k+1}, \beta_{n-k+1} \in \mathbb{Z}_+^n$ satisfying $\sum_{i=1}^{n-k+1} \alpha_i = \alpha$ and $\sum_{i=1}^{n-k+1} \beta_i = \beta$ (all empty products are interpreted as 1).

- Base Case: First suppose $n = 0$. Then the result is trivial. For notational simplicity we will also prove the case $n = 1$ directly. In this case, we have by the chain rule
$$(\partial_x^\alpha \partial_\xi^\beta)(F(a(x,\xi)) = (F^{(1)}(a(x,\xi)))(\partial_x^\alpha \partial_\xi^\beta a(x,\xi))$$
which is of the desired form.
- Induction Step: Assume that the claim holds for all $n \leq j \geq 1$. First we consider x derivatives. Consider a term of the form (42) in the expression for $(\partial_x^\alpha \partial_\xi^\beta)(F(a(x,\xi)))$. Let $\tilde{\alpha}, \tilde{\beta} \in \mathbb{Z}_+^n$ be multi-indices with $|\tilde{\alpha}| + |\tilde{\beta}| = 1$. Then by the chain and product rules,

$$\begin{aligned}
&(\partial_x^{\tilde{\alpha}} \partial_\xi^{\tilde{\beta}})(F^{(k)}(a(x,\xi)) \prod_{i=1}^{k} (\partial_x^{\alpha_i} \partial_\xi^{\beta_i} a)(x,\xi)) \\
=&(F^{(k+1)}(a(x,\xi)))(\partial_x^{\tilde{\alpha}} \partial_\xi^{\tilde{\beta}} a(x,\xi)) \prod_{i=1}^{k} (\partial_x^{\alpha_i} \partial_\xi^{\beta_i} a)(x,\xi)) + \\
&(F^{(k)}(a(x,\xi))) \sum_{m=1}^{k} (\prod_{i=1}^{m-1} (\partial_x^{\alpha_i} \partial_\xi^{\beta_i} a)(x,\xi)(\partial_x^{\alpha_m+\tilde{\alpha}} \partial_\xi^{\beta_m+\tilde{\beta}} a)(x,\xi) \prod_{i=m+1}^{k} (\partial_x^{\alpha_i} \partial_\xi^{\beta_i} a)(x,\xi))
\end{aligned}$$

  which is a linear combination of terms of the form (42), as desired.

It remains to show that terms of the form (42) are in $T^{-|\beta|}$. By (40), $F^{(k)}(a(x,\xi)))$ is bounded. Moreover, since $a \in S^0$, we have that $(\partial_x^{\alpha_i} \partial_\xi^{\beta_i} a)(x,\xi) \in S^{-|\beta_i|}$. Thus, $\prod_{i=1}^{k} (\partial_x^{\alpha_i} \partial_\xi^{\beta_i} a)(x,\xi) \in S^{-|\beta|} \subset T^{-|\beta|}$, and so terms of the form (42) are in $T^{-|\beta|}$ as well. Thus (41) has been established. □

Proof. (of Lemma 4.2) We will define a sequence b_j which approximates a_j and is such that $\sum_j b_j$ converges. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be satisfy on $\phi_{B_1(0)} \equiv 1$ and $\phi_{(\bar{B}_2(0)^c)} \equiv 0$. Let $c_j \in (0,1)$ be sequences with $\lim_{j\to\infty} c_j = 0$, and define $b_j(x,\xi) = (1-\phi(c_j\xi))a_j(x,\xi)$. As $b_j - a_j$ has compact support, $b_j - a_j \in S^{-\infty}$, and so

$b_j \in S^{m-j}$.

Now, if $|\xi| \leq 2/c_j$, then by definition of λ we have $\lambda(\xi)c_j \leq \sqrt{5}$. Hence

$$|\partial_x^\alpha \partial_\xi^\beta b_j| \leq \sum_{|\gamma|\leq\beta} C_\gamma c_j^{|\gamma|} |\partial_x^\alpha \partial_\xi^{\beta-\alpha} a_j| \leq C_{\alpha\beta}^j \lambda^{m-j-|\beta|}$$

for some constants $C_{\alpha\beta}^j$. A similar result holds for $\xi \geq 2/c_j$ since $b_j = a_j$ there. Since $1 \leq c_j|\xi|$ in supp $(1-\phi) \subset$ supp b_j, the estimate can be improved:

$$|\partial_x^\alpha \partial_\xi^\beta b_j| \leq c_j \lambda |\partial_x^\alpha \partial_\xi^\beta b_j| \leq c_j C_{\alpha\beta}^j \lambda^{m+1-j-|\beta|}$$

Thus if $c_j \leq \min \{1/C_{\alpha\beta}^j \lambda^{m+1-j-|\beta|}\}$, then $|\lambda^{|\beta|-m} \partial_x^\alpha \partial_\xi^\beta b_j| \leq \lambda^{1-j}$ when $|\alpha+\beta| \leq j$. Since $c_j \to 0$, we have

$$a(x,\xi) := \sum_{j\geq 0} b_j(x,\xi) < \infty$$

near any fixed ξ_0 and so the sum defines a function $a \in C^\infty$. If $k \in \mathbb{Z}_+$ and $\alpha, \beta \in \mathbb{Z}_+^n$ are fixed, and we take $N = \max(|\alpha+\beta|, k+1)$, then we can write

$$a - \sum_{j<k} a_j = \sum_{j<k} (b_j - a_j) + \sum_{k\leq j<N} b_j + \sum_{j\geq N} b_j$$

The sums $\sum_{j<k}$ and $\sum_{k\leq j<N}$ are in S^{m-k} as finite sums of terms in S^{m-k}. So consider the sum $\sum_{j\geq N}$, then

$$\begin{aligned}
|\lambda^{|\beta|-(m-k)} \partial_x^\alpha \partial_\xi^\beta \sum_{j\geq N} b_j| &\leq \sum_{j\geq N} |\lambda^{|\beta|-m+k} \partial_x^\alpha \partial_\xi^\beta b_j| \\
&\leq \sum_{j\geq k+1} \lambda^{k+1-j} \\
&\leq \frac{\sqrt{2}}{\sqrt{2}-1}
\end{aligned}$$

since $|\alpha+\beta| \leq j$ and $\lambda(\xi) \geq \sqrt{2}$ on supp b_j. Thus we have $a - \sum_{j<k} a_j \in S^{m-k}$, which for $k = 0$ implies that $a \in S^m$. The property of the supports follows by construction. □

Lemma A.1. *Suppose q is a nondegenerate real quadratic form on $\mathbb{R}^n$ and $\chi \in C_0^\infty$ with $\chi = 0$ near 0. Then for all $N \in \mathbb{Z}_+$,*

$$\left| \int e^{i\mu^2 q(y)} b(\mu y) \chi(y) dy \right| \leq C_N \mu^{-N} \sup_{y \in \text{ supp } \chi, |\alpha|\leq N} |(\partial^\alpha b)(\mu y)|$$

where C_N is independent of $\mu \geq 1$ and $b \in C^\infty$.

Proof. There is a linear change of variables so that $q(y) = |y'|^2 - |y''|^2$ with $y = (y', y'')$. Then the operator $L = (1/2|y|^2)(\langle y', \partial' \rangle - \langle y'', \partial'' \rangle)$ is well defined on supp χ with C^∞ coefficients and satisfies $Lq = 1$. Integrating by parts involves the transpose of L, ${}^tL$, which is given by ${}^tL = \langle \partial'', y''/2|y|^2 \rangle - \langle \partial', y'/2|y|^2 \rangle$. Note that ${}^tL$ is also a first-order differential operator with C^∞ coefficients. Integrating by parts N times gives

$$\begin{aligned}\int e^{i\mu^2 q(y)} b(\mu y)\chi(y)dy &= (i\mu^2)^{-N}\int (L^N e^{i\mu^2 q(y)})b(\mu y)\chi(y)dy \\ &= (i\mu^2)^{-N}\int e^{i\mu^2 q(y)}({}^tL)^N(b(\mu y)\chi(y))dy \\ &= (i\mu^2)^{-N}\int e^{i\mu^2 q(y)} c_{\mu,N}(y)dy\end{aligned}$$

where $c_{\mu,N}$ is a linear combination with C^∞ coefficients of terms of the form $\mu^{|\alpha|}((\partial^\alpha b)(\mu y))(\partial^\beta\chi(y))$ for $|\alpha+\beta|\le N$. As supp χ is compact, the result follows. □

Proof. (of Lemma 4.3) If $a\in L^1$ this follows immediately from the Lebesgue Dominated Convergence Theorem.

For the general case, take $\psi\in C_0^\infty(\mathbb{R}^n)$ such that $\psi=1$ on B_1 and supp $\psi\subset\bar{B}_2$. Define $I_j=\int e^{iq(x)}a(x)\phi(\epsilon x)\psi(2^{-j}x)dx$. First I will show that $\lim_{j\to\infty} I_j$ exists and is equal to $\lim_{\epsilon\to 0} e^{iq(x)}a(x)\phi(\epsilon x)dx$, and that these limits exist for any $\phi\in\mathcal{S}$ and is independent of choice of ϕ. Now, since

$$\int e^{iq(x)}a(x)\phi(\epsilon x)dx = \lim_{j\to\infty}\int e^{iq(x)}a(x)\phi(\epsilon x)\phi(2^{-j}x)dx$$

for any fixed $\epsilon>0$ by dominated donvergence, we define

$$I_j(\epsilon)=\int e^{iq(x)}a(x)(1-\phi(\epsilon x))\psi(2^{-j}x)dx$$

and show that $\lim_{j\to\infty} I_j$ exists and that $\lim_{j\to\infty} I_j(\epsilon)=0(\epsilon)$ (look into this). First take $y=2^{-j}x$. Then

$$\begin{aligned}I_j-I_{j-1} &= \int e^{iq(x)}a(x)(\psi(2^{-j}x)-\psi(2^{1-j}x))dx \\ &= \int e^{i2^{2j}q(y)}a(2^jy)(\psi(y)-\psi(2y))2^{jn}dy\end{aligned}$$

and similarly

$$I_j(\epsilon)-I_{j-1}(\epsilon)=\int e^{i2^{2j}q(y)}a(2^jy)(1-\phi(\epsilon 2^jy))(\psi(y)-\psi(2y))2^{jn}dy$$

Now let $\chi(y)=\psi(y)-\psi(2y)$. Then $\chi\in C_0^\infty$ and supp $\chi\subset\{y:1/2\le|y|\le 2\}$. In addition, $y\in$ supp χ implies

$$\begin{aligned}|(\partial^\alpha a)(2^jy)| &\le |||a|||_{|\alpha|}(1+2^{2j}|y|^2)^{m/2} \\ &\le |||a|||_{|\alpha|}2^{m(j+2)}\end{aligned}$$

and similarly $|1-\phi(\epsilon 2^jy)|\le|\epsilon 2^jy|\sup_{\mathbb{R}^n}|\phi'|\le\epsilon C2^j$ implies

$$|(\partial^\alpha b_\epsilon)(2^jy)|\le\epsilon C2^{(m+1)j}$$

where $b_\epsilon(x)=(1-\phi(\epsilon x))a(x)$ and C does not depend on ϵ or j. Taking $\mu=2^j$ and $N=m+n+1$ (and $N=M+n+2$, respectively) in Lemma A.1, we have

$$\begin{aligned}|I_j-I_{j-1}| &\le C_{q,m}2^{-j}|||a|||_{m+n+1} \\ |I_j(\epsilon)-I_{j-1}(\epsilon)| &\le \epsilon C2^{-j}\end{aligned}$$

and the result follows. □

Proof. (of Lemma 4.4) Since $S^l \subset S^m$ when $l \leq m$, we assume without loss of generality that $a \in S^{2m}$ for some $m \in \mathbb{Z}_+$. Then since $\phi \in \mathcal{S}$ we have $\hat{\phi} \in \mathcal{S}$, and so

$$|a(x,D)\phi(x)| \leq (2\pi)^{-n} \int \|\lambda^{-2m} a\|_\infty \|\lambda^{2m+2n}\hat{\phi}\|_\infty \lambda^{-2n}(\xi) d\xi$$

Thus $a(x,D)\phi$ is bounded and $\|a(x,D)\phi\|_\infty \leq C|\hat{\phi}|_{2m+2n}$. Moreover, $\|a(x,D)\phi\| \leq C_0|\phi|_N$ with $N = 2m+4m$ be the continuity of the Fourier Transform. In addition, differentiating under the integral gives

$$\partial_j(a(x,D)\phi(x)) = a(x,D)(\partial_j\phi)(x) + (\partial_{x_j}a)(x,D)\phi(x)$$

Integrating by parts, we see

$$x_j(a(x,D)\phi(x)) = a(x,D)(x_j\phi)(x) + i(\partial_{\xi_j}a)(x,D)\phi(x)$$

Hence

$$x^\alpha \partial^\beta(a(x,D)\phi(x))$$

can be written as a linear combination of terms

$$(\partial_x^\gamma \partial_\xi^\delta a)(x,D)(x^{\alpha-\delta}\partial^{\beta-\gamma}\phi)(x)$$

and so $a(x,D)\phi \in \mathcal{S}$ with $|a(x,D)\phi|_k \leq C_k|\phi|_{k+N}$. □

Proof. (of Lemma 4.5) This proof essentially consists of checking that the integrals in the statement of the lemma are indeed oscillatory integrals, and then letting $\epsilon \to 0$ as in the definition (when they are actual integrals).

(i) This follows from the change of variables $x = Ay$ in the integral

$$\int e^{iq(x)} a(x)\phi(\epsilon x) dx$$

since $\psi(y) = \phi(Ay) \in \mathcal{S}$ satisfies $\psi(0) = \phi(0) = 1$ and since $b(y) = |\det A| a(Ay)$ is an amplitude of order m.

(ii) Integrations by parts in the right side of the given equation with the added factor $\phi(\epsilon x)$ give a factor

$$\partial^\alpha(\phi(\epsilon x)b(x)) = \sum_\beta \binom{\alpha}{\beta} \epsilon^{|\beta|}(\partial^\beta\phi)(\epsilon x)\partial^{\alpha-\beta}b(x)$$

and for $\beta \neq 0$, the $\epsilon^{|\beta|}$ gives zero as $\epsilon \to 0$, while for $\beta = 0$ we get the left hand side.

(iii) Recall the proof of Lemma 4.3. We considered the integrals

$$I_j(y) = \int e^{iq(x)} a(x,y)\psi(2^{-j}x) dx$$

which satisfy $\partial_y^\alpha I_j(y) = \int e^{iq(x)} \partial_y^\alpha a(x,y)\psi(2^{-j}x)dx$ because of the absolute convergence of the factor $\psi(2^{-j}x)$. Since for $|z| \leq 2$

$$\begin{aligned}|\partial_x^\beta \partial_y^\alpha a(\mu z, y)| &\leq C_{\alpha\beta}(1 + |\mu z|^2 + |y|^2)^{m/2} \\ &\leq C_{\alpha\beta} 5^{m/2} \mu^m (1 + |y|^2)^{m/2}\end{aligned}$$

Lemma A.1 gives the estimates

$$|\partial_y^\alpha I_j(y) - \partial_y^\alpha I_{j-1}(y)| \leq C_\alpha 2^{-j}(1 + |y|^2)^{m/2}$$

which imply uniform convergence on every compact set for the sequence $\partial_y^\alpha I_j(y)$. Hence the limit $I(y)$ of the sequence $I_j(y)$ is in $A^m(\mathbb{R}^p)$ and satisfies $\partial_y^\alpha I(y) = \lim_{j\to\infty} \partial_y^\alpha I_j(y)$.

(iv) The estimates in the previous part show that

$$|\partial_y^\alpha (I(y) - I_j(y))| \leq C_\alpha 2^{-J}(1_|y|^2)^{m/2}$$

so that the functions $b_j(y) = \psi(2^{-j}y)(I(y) - I_j(y))$ satisfy $b_j \in A^m(\mathbb{R}^p)$ with $|||b_j|||_{m+p+1} \leq C_0 2^{-j}$. So we can write

$$\int e^{ir(y)}(\int e^{iq(x)}a(x,y)dx)dy = \lim_{j\to\infty} \int e^{ir(y)} I(y)\psi(2^{-J}y)dy$$

and

$$\int e^{ir(y)} I(y)\psi(2^{-j}y)dy = \int e^{ir(y)} I_j(y)\psi(2^{-j}y)dy + \int e^{ir(y)} b_j(y)dy$$

Thus the property follows since

$$\lim_{j\to\infty} \int e^{ir(y)} I_j(y)\psi(2^{-j}y)dy = \int e^{i(q(x)+r(y))}a(x,y)dxdy$$

and

$$|\int e^{ir(y)} b_j(y)dy| \leq C_{r,m}|||b_j|||_{m+p+1} \leq C_{r,m}C_0 2^{-j}$$

□

Lemma A.2. *(Peetre's Inequality) For any $s \in \mathbb{R}$ and all $\xi, \eta \in \mathbb{R}^n$,*

$$\lambda^s(\xi) \leq 2^{|s|}\lambda^{|s|}(\xi - \eta)\lambda^s(\eta)$$

Proof. From the triangle inequality,

$$(1 + |\xi|) \leq (1 + |\xi - \eta| + |\eta|) \leq (1 + |\xi - \eta|)(1 + |\eta|)$$

Hence

$$\lambda^2(\xi) \leq (1 + |\xi|)^2 \leq (1 + |\xi - \eta|)^2(1 + |\eta|^2)$$

We also have

$$(1 + |\eta|)^2 \leq (1 + |\eta|)^2 + (1 - |\eta|)^2 = 2\lambda^2(\eta)$$
$$(1 + |\xi - \eta|)^2 \leq (1 + |\xi - \eta|)^2 + (1 - |\xi - \eta|)^2 = 2\lambda^2(\xi - \eta)$$

Thus

$$\lambda^2(\xi) \leq 2^2\lambda^2(\xi - \eta)\lambda^2(\eta)$$

When $s \geq 0$ the result follows by taking the power $s/2$. When $s < 0$, switching ξ and η gives

$$\lambda^{-s}(\eta) \leq 2^{-s}\lambda^{-s}(\eta - \xi)\lambda^{-s}(\xi)$$

or

$$\lambda^s(\xi) \leq 2^{-s}\lambda^{-s}(\xi - \eta)\lambda^s(\eta)$$

as desired. □

Lemma A.3. *We have the following.*

(*i*) *If $a \in A^m(\mathbb{R}^n)$, then*

$$(2\pi)^{-\eta} \int e^{-i\langle y,\eta\rangle}a(y)dyd\eta = (2\pi)^{-n} \int e^{-i\langle y,\eta\rangle}a(\eta)dyd\eta = a(0)$$

(ii) If $\alpha, \beta \in \mathbb{Z}_+^n$, we have

$$(2\pi)^{-n} \int e^{-i\langle y,\eta\rangle} \frac{y^\alpha}{\alpha!} \frac{\eta^\beta}{\beta!} dy d\eta = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ (-i)^{|\alpha|}/\alpha! & \text{if } \alpha = \beta \end{cases}$$

Proof. First note that $\langle y, \eta\rangle$ is nondegenerate as a quadratic form on $\mathbb{R}^{2n} \cong \mathbb{R}^n \times \mathbb{R}^n$. To see this, recall that a quadratic form $q(x)$ is said to be nondegenerate if the associated bilinear form $b(x, y)$ defined by $b(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y))$ is nondegenerate. In the case when when $q(x) = q(x_1, x_2) = x_1 \cdot x_2$ for $x = (x_1, x_2) \in \mathbb{R}^{2n}$, we have

$$\begin{aligned} b(x, y) &= \frac{1}{2}(q(x + y) - q(x) - q(y)) \\ &= \frac{1}{2}((x_1 + y_1) \cdot (x_2 + y_2) - x_1 \cdot x_2 - y_1 \cdot y_2) \\ &= \frac{1}{2}(x_1 \cdot y_2 - x_2 \cdot y_1) \end{aligned}$$

which is clearly nondegenerate.

(i) The first equality follows by symmetry and Fubini's Theorem. For the second equality, let $\phi \in \mathcal{S}$ with $\phi(0) = 1$. So by definition of oscillatory integrals,

$$\int e^{-i\langle y,\eta\rangle} a(\eta) dy d\eta = \lim_{\epsilon \to 0} \int e^{-i\langle y,\eta} a(\eta)\phi(\epsilon y)\phi(\epsilon\eta) dy d\eta$$

Let $z = \epsilon y, \zeta = \eta/\epsilon$, and integrate in z to get

$$\int e^{-i\langle z,\zeta\rangle} a(\epsilon\zeta)\phi(z)\phi(\epsilon^2\zeta) dz d\zeta = \int \hat{\phi}(\zeta) a(\epsilon\zeta)\phi(\epsilon^2\zeta) d\zeta$$

When $\epsilon < 1$, $|\hat{\phi}(\zeta)a(\epsilon\zeta)\phi(\epsilon^2\zeta)| \leq |\hat{\phi}(\zeta)|||a|||_0(1 + |\zeta|^2)^{m/2}|\phi|_0$. This is integrable, so by dominated convergence,

$$(2\pi)^{-n} \int e^{-i\langle y,\eta\rangle} a(\eta) dy d\eta = (2\pi)^{-n} \int \hat{\phi}(\zeta) a(0) d\zeta = \phi(0)a(0) = a(0)$$

(ii) When $\alpha, \beta \in \mathbb{Z}_+^n$, we have $y^\alpha e^{-i\langle y,\eta\rangle} = (-D_\eta)^\alpha e^{-i\langle y,\eta\rangle}$. Thus

$$(2\pi)^{-n} \int e^{-i\langle y,\eta\rangle} \frac{y^\alpha}{\alpha!} \frac{\eta^\beta}{\beta!} dy d\eta = (2\pi)^{-n} \int e^{-i\langle y,\eta\rangle} \frac{D_\eta^\alpha}{\alpha!} \left(\frac{\eta^\beta}{\beta!}\right) dy d\eta$$

The function $a(\eta) = \frac{D_\eta^\alpha}{\alpha!}\left(\frac{\eta^\beta}{\beta!}\right) = \frac{(-i)^{|\alpha|}}{\beta!}\binom{\beta}{\alpha}\eta^{\beta-\alpha}$ satisfies $a(0) = 0$ when $\beta \neq \alpha$ and $a(0) = (-i)^{|\alpha|}/\alpha!$ if $\beta = \alpha$, so (ii) follows from (i).

□

Proof. (of Lemma 4.6) From the beginning of the proof of Lemma A.3, we see that the quadratic form $\langle y, \eta\rangle$ is nondegenerate. Now, Peetre's inequality gives that

$b_{x,\xi}(y,\eta) := \bar{a}(x-y,\xi-\eta)$ is an amplitude when x,ξ are fixed:

$$\begin{aligned}|\partial_y^\alpha \partial_\eta^\beta \bar{a}(x-y,\xi-\eta)| &\leq C_{\alpha\beta}\lambda^{m-|\beta|}(\xi-\eta) \\ &\leq C_{\alpha\beta}\lambda^{m}(\xi-\eta) \\ &\leq C_{\alpha\beta}2^{|m|}\lambda^{|m|}(\eta)\lambda^m(\xi) \\ &\leq C_{\alpha\beta}2^{|m|}\lambda^m(\xi)(1+|y|^2+|\eta|^2)^{|m|/2}\end{aligned}$$

for all $\alpha,\beta \in \mathbb{Z}_+^n$, and so $b_{x,\xi} \in A^{|m|}(\mathbb{R}^{2n})$ with $|||b_{x,\xi}|||_{|m|+2n+1} \leq C_0\lambda^m(\xi)$. By Lemma 4.3, $\lambda^{-m}(\xi)a^*(x,\xi)$ is bounded. Also, since $\partial_x^\alpha\partial_\xi^\beta(a^*) = (\partial_x^\alpha\partial_\xi^\beta a)^*$ and $\partial_x^\alpha\partial_\xi^\beta a \in S^{m-|\beta|}$, $\lambda^{|\beta|-m}\partial_x^\alpha\partial_\xi^\beta a^*$ is bounded for any $\alpha,\beta\in\mathbb{Z}_+^n$ by the same argument. Hence $a^* \in S^m$.

Next we consider $a\#b$. The function $c_{x,\xi}(y,\eta) := a(x,\xi-\eta)b(x-y,\xi)$ is also an amplitude–if we fix (x,ξ), we have:

$$\begin{aligned}|\partial_y^\alpha\partial_\eta^\beta a(x,\xi-\eta)b(x-y,\xi)| &= |\partial_\eta^\beta a(x,\xi-\eta)||\partial_y^\alpha b(x-y,\xi)| \\ &\leq C_\beta(1+|\xi-\eta|^2)^{\frac{m-|\beta|}{2}}C_\alpha(1+|\xi|^2)^{\frac{l}{2}} \\ &\leq C_\alpha\beta\lambda^{m-|\beta|}(\xi-\eta) \\ &\leq C_{\alpha\beta}2^{|m|}\lambda^m(\xi)(1+|y|^2+|\eta|^2)^{|m|/2}\end{aligned}$$

where the last line follows from the calculation for a^*. Thus $c_{x,\xi}\in A^{|m|}(\mathbb{R}^{2n})$ and $|||c_{x,\xi}|||_{|m|+2n+1} \leq C_0\lambda^{m+l}(\xi)$. Hence, as above, we see that $\lambda^{-m-l}(\xi)a\#b(x,\xi)$ is bounded. By the product rule,

$$\partial_x^\alpha\partial_\xi^\beta(a\#b)(x,\xi) = \sum_{(\gamma,\delta)\in\mathbb{Z}_+^{2n}} \binom{(\alpha,\beta)}{(\gamma,\delta)}(\partial_x^\gamma\partial_\xi^\delta a)\#(\partial_x^{\alpha-\gamma}\partial_\xi^{\beta-\delta}b)$$

Now, $\partial_x^\gamma\partial_\xi^\delta a \in S^{m-|\delta|}$ and $\partial_x^{\alpha-\gamma}\partial_\xi^{\beta-\delta}b\in S^{l-|\beta-\delta|}$ for all γ,δ. Hence

$$|(\partial_x^\gamma\partial_\xi^\delta a)\#(\partial_x^{\alpha-\gamma}\partial_\xi^{\beta-\delta}b)| \leq C\lambda^{m+l-|\beta|}$$

and so $\lambda^{|\beta|-m-l}\partial_x^\alpha\partial_\xi^\beta(a\#b)(x,\xi)$ is bounded for any $\alpha,\beta\in\mathbb{Z}_+^n$. So $a\#b\in S^{m+l}$.

The asymptotic expansions are proved using Taylor's formula:

$$\begin{aligned}\bar{a}(x-y,\xi-\eta) &= \sum_{|\alpha+\beta|<2k}\frac{(-y)^\alpha}{\alpha!}\frac{(-\eta)^\beta}{\beta!}\partial_x^\alpha\partial_\xi^\beta\bar{a}(x,\xi)+r_k(x,\xi,y,\eta) \\ r_k(x,\xi,y,\eta) &= \sum_{|\alpha+\beta|=2k} 2k\frac{(-y)^\alpha}{\alpha!}\frac{(-\eta)^\beta}{\beta!}r_{\alpha\beta}(x,\xi,y,\eta) \\ r_{\alpha\beta}(x,\xi,y,y,\eta) &= \int_0^1 (1-t)^{2k-1}\partial_x^\alpha\partial_\xi^\beta\bar{a}(x-ty,\xi-t\eta)dt\end{aligned}$$

The terms with $|\alpha+\beta|<2k$ give after integration the terms of the expansion in view of Lemma A.3(ii). Note that $r_k(y,\eta)\in A^{|m|+2k}$, so we can integrate by parts as in Lemma A.3(ii):

$$
\begin{aligned}
&\int e^{-i\langle y,\eta\rangle}\frac{(-y)^\alpha}{\alpha!}\frac{(\eta)^\beta}{\beta!}r_{\alpha\beta}(x,\xi,y,\eta)dyd\eta\\
=&\frac{1}{\alpha}\int\frac{(-\eta)^\beta}{\beta!}r_{\alpha\beta}(x,\xi,y,\eta)D_\eta^\alpha(e^{-i\langle y,\eta\rangle})dyd\eta\\
=&\frac{1}{\alpha!}\int e^{-i\langle y,\eta\rangle}\sum_\gamma\binom{\alpha}{\gamma}((-D_\eta)^\gamma\frac{(-\eta)^\beta}{\beta!})((-D_n)^{\alpha-\gamma}r_{\alpha\beta}(x,\xi,y,\eta))dyd\eta\\
=&\sum_\gamma\frac{(-i)^{|\gamma|}\gamma!}{\alpha!\beta!}\binom{\alpha}{\gamma}\binom{\beta}{\gamma}\int e^{-i\langle y,\eta\rangle}(-\eta)^{\beta-\gamma}(-D_\eta^{\alpha-\gamma}r_{\alpha\beta}(x,\xi,y,\eta)dyd\eta\\
=&\sum_\gamma\frac{(-i)^{|\gamma|}\gamma!}{\alpha!\beta!}\binom{\alpha}{\gamma}\binom{\beta}{\gamma}\int e^{-i\langle y,\eta\rangle}(-D_y)^{\beta-\gamma}(-D_\eta)^{\alpha-\gamma}r_{\alpha\beta}(x,\xi,y,\eta)dyd\eta
\end{aligned}
$$

after a second integration by parts. By definition of $r_{\alpha\beta}$,

$$
\begin{aligned}
&(-D_y)^{\beta-\gamma}(-D_\eta)^{\alpha-\gamma}r_{\alpha\beta}(x,\xi,y,\eta)\\
=&\int_0^1(1-t)^{2k-1}(-it)^{2k-2|\gamma|}\partial_x^{\alpha+\beta-\gamma}\partial_\xi^{\alpha+\beta-\gamma}\bar{a}(x-ty,\xi-t\eta)dt
\end{aligned}
$$

$\gamma\le\alpha$ and $\gamma\le\beta$, so $|\gamma|\le k$ and $|\alpha+\beta-\gamma|\ge k$. Thus $\partial_x^{\alpha+\beta-\gamma}\partial_\xi^{\alpha+\beta-\gamma}\bar{a}\in S^{m-k}$. Hence the equations above can be summarized by

$$
\int e^{-i\langle y,\eta\rangle}r_k(x,\xi,y,\eta)dyd\eta=\int e^{-i\langle y,\eta\rangle}s_k(x,\xi,y,\eta)dyd\eta
$$

where the amplitude $s_k\in A^{|m-k|}$ with $|||s_k|||_{|m-k|+2n+1}\le C_k\lambda^{m-k}(\xi)$. So

$$
\lambda^{k-m}(\xi)\int e^{-i\langle y,\eta\rangle}r_k(x,\xi,y,\eta)dyd\eta
$$

is bounded. Then, arguing as above, $\int e^{-i\langle y,\eta\rangle}r_k(x,\xi,y,\eta)dyd\eta\in S^{m-k}$ since $\partial_x^\alpha\partial_\xi^\beta r_k$ is the rest of index $2k$ in the Taylor expansion of $\partial_x^\alpha\partial_\xi^\beta\bar{a}(x-y,\xi-\eta)$, and $\partial_x^\alpha\partial_\xi^\beta\bar{a}\in S^{m-|\beta|}$.

The argument asymptotic expansion for $a\#b$ is the same as the argument for the asymptotic expansion for a^*, verbatim, except with $\bar{a}(x-ty,\xi-t\eta)$ replaced by $a(x,\xi-t\eta)b(x-ty,\xi)$ and S^m replaced with S^{m+l}. □

Proof. (of Lemma 4.7)

(i) We have

$$
\begin{aligned}
&(a^*)^*(x,\xi)\\
=&(2\pi)^{-2n}\int e^{-i\langle z,\zeta\rangle}(\overline{\int e^{-i\langle y,\eta\rangle}\bar{a}(x-z-y,\xi-\zeta-\eta)dyd\eta})dzd\zeta\\
=&(2\pi)^{-2n}\int e^{i(\langle y,\eta\rangle-\langle z,\zeta\rangle)}a(x-z-y,\xi-\zeta-\eta)dyd\eta dzd\zeta
\end{aligned}
$$

Make the change of variables $Y=-y$, $H=\eta+\xi$, $Z=z+y$, $\mathcal{Z}=\zeta$, for which $\langle y,\eta\rangle-\langle z,\eta\rangle=-\langle Y,H\rangle-\langle Z,\mathcal{Z}\rangle$ and $dyd\eta dzd\zeta=dYdHdZd\mathcal{Z}$.

Then

$$
\begin{aligned}
&(a^*)^*(x,\xi)\\
=&(2\pi)^{-2n}\int e^{-e(\langle Y,H\rangle+\langle Z,\mathcal{Z}\rangle)}a(x-Z,\xi-H)dYdYdZd\mathcal{Z}\\
=&(2\pi)^{-2n}\int e^{-i\langle Z,\mathcal{Z}\rangle}(\int e^{-i\langle Y,H\rangle}a(x-Z,\xi-H)dYdH)dZd\mathcal{Z}\\
=&(2\pi)^{-n}\int e^{-i\langle Z,\mathcal{Z}\rangle}a(x-Z,\xi)dZd\mathcal{Z}\\
=&a(x,\xi)
\end{aligned}
$$

where the last two equalities are consequences of the fact that

(43) $$(2\pi)^{-n}\int e^{-i\langle y,\eta\rangle}a(y)dyd\eta=(2\pi)^{-n}\int e^{-\langle y,\eta\rangle}a(\eta)dyd\eta=a(0)$$

This can be seen as follows. First note that the quadratic form on $\mathbb{R}^n$ given by $(y,\eta)\to\langle y,\eta\rangle$ is nondegenerate (by the proof of Lemma A.3). Moreover, the polynomial $y^\alpha\eta^\beta$ is in $A^{|\alpha+\beta|}$ so that the integrals in equation (43) are indeed osciallatory integrals. The first equality follows by switching y and η. Next, take $\phi\in\mathcal{S}$ such that $\phi(0)=1$. By definition, we have

$$\int e^{-i\langle y,\eta\rangle}a(\eta)dyd\eta=\lim_{\epsilon\to 0}\int e^{-i\langle y,\eta\rangle}a(\eta)\phi(\epsilon y)\phi(\epsilon\eta)dyd\eta$$

Making the change of variables $\epsilon y=z$, $\epsilon\xi=\zeta$ and then integrating in z we get

$$\int e^{-i\langle z,\zeta\rangle}a(\epsilon\zeta)\phi(z)\phi(\epsilon^2\zeta)dzd\zeta=\int\hat{\phi}(\zeta)a(\epsilon\zeta)\phi(\epsilon^2\zeta)d\zeta$$

When $\epsilon<1$, we have

$$|\hat{\phi}(\zeta)a(\epsilon\zeta)\phi(\epsilon^2\zeta)|\leq|\hat{\phi}(\zeta)|\|\|a\||_0(1+|\zeta|^2)^{m/2}|\phi|_0$$

which is integral. Hence by dominated convergence,

$$
\begin{aligned}
(2\pi)^{-n}\int e^{-\langle y,\eta\rangle}a(\eta)dyd\eta&=(2\pi)^{-n}\int\hat{\phi}(\zeta)a(0)d\zeta\\
&=\phi(0)a(0)\\
&=a(0)
\end{aligned}
$$

and we are finished.

(ii) The proof of Lemma 4.6 with $k=m+1$ gives

$$a\#b=\sum_{|a|\leq m}(1/\alpha!)\partial_\xi^\alpha aD_x^\alpha b$$

for any $b\in S^l$, and the result follows.

(iii) Write

$$
\begin{aligned}
&a\#(b\#c)(x,\xi)\\
=&(2\pi)^{-2n}\int e^{-i\langle y,\eta\rangle}a(x,\xi-\eta)(\int e^{-\langle z,\zeta\rangle}b(x-y,\xi-\zeta)c(x-y-z,\xi)dzd\zeta)dyd\eta\\
=&(2\pi)^{-2n}\int e^{-i(\langle y,\eta\rangle+\langle z,\zeta\rangle)}a(x,\xi-\eta)b(x-y,\xi-\zeta)c(x-y-z,\xi)dyd\eta dzd\zeta
\end{aligned}
$$

Hence we have

$$\begin{aligned}
&(a\#b)\#c(x,\xi)\\
=&(2\pi)^{-2n}\int e^{-i\langle Z,\mathcal{Z}\rangle}(\int e^{-i\langle Y,H\rangle}a(x,\xi-\mathcal{Z}-H)b(x-Y,\xi-\mathcal{Z})dydH)c(x-Z,\xi)dZd\mathcal{Z}\\
=&(2\pi)^{-2n}\int e^{-i(\langle Y,H\rangle+\langle Z,\mathcal{Z}\rangle)}a(x,\xi-\mathcal{Z}-H)b(x-Y,\xi-\mathcal{Z})c(x-Z,\xi)dYdHdZd\mathcal{Z}
\end{aligned}$$

These two quantities are equal through the change of variables $y = Y$, $\eta = H + \mathcal{Z}$, $z = Z - Y$, and $\zeta = \mathcal{Z}$.

(iv) Next, we have

$$\begin{aligned}
&b^*\#a^*(x,\xi)\\
=&(2\pi)^{-3n}\int e^{-i\langle t,\tau\rangle}(\int e^{-i\langle z,\zeta\rangle}\bar{b}(x-z,\xi-\tau-\zeta dzd\zeta)(\int e^{-i\langle y,\eta\rangle}\bar{a}(x-t-y,\xi-\eta)dyd\eta)dtd\tau\\
=&(2\pi)^{-3n}\int e^{-i(\langle y,\eta\rangle+\langle z,\zeta\rangle+\langle t,\tau\rangle)}\bar{a}(x-t-y,\xi-\eta)\bar{b}(x-z,\xi-\tau-\zeta)dyd\eta dzd\zeta dtd\tau\\
=&(2\pi)^{-3n}\int e^{-i(-\langle Y,H\rangle+\langle Z,\mathcal{Z}\rangle+\langle X,\Xi\rangle)}\bar{a}(x-Z,\xi-\mathcal{Z}-H)\bar{(}x-Z_Y,\xi-\mathcal{Z})dYdHdZd\mathcal{Z}dXd\Xi\\
=&(2\pi)^{-2n}\int e^{-i\langle Z,\mathcal{Z}\rangle}\overline{(\int e^{-i\langle Y,H\rangle}a(x-Z,\xi-\mathcal{Z}-H)b(x-Z-Y,\xi-\mathcal{Z}dYdH)}dZd\mathcal{Z}
\end{aligned}$$

after a change of variables ($Y = z - t - y$, $H = \eta - \tau - \xi$, $Z = t + y$, $\mathcal{Z} = \tau + \xi$, $X = z - t$, $\Xi = \eta - \tau$) and the last equality follows from integration in (X, Ξ) and Lemma A.3(i). Thus the result follows since

$$a\#b(x-Z,\xi-\mathcal{Z}) = (2\pi)^{-n}\int e^{-i\langle Y,H\rangle}a(x-Z,\xi-\mathcal{Z}-H)b(x-Z-Y,\xi-\mathcal{Z})dYdH$$

(v) ($I_0^* = (a^*(x,D)\phi,\psi)$ is equal to the oscillatory integral

$$\begin{aligned}
I_0^* &= (2\pi)^{-2n}\int e^{i\langle x,\xi\rangle}(\int e^{-i\langle y,\eta\rangle}\bar{a}(x-y,\xi-\eta)dyd\eta)\hat{\phi}(\xi)\bar{\psi}(x)dxd\xi\\
&= (2\pi)^{-2n}\int e^{i(\langle x,\xi\rangle-\langle x-z,\xi-\zeta\rangle)}\bar{a}(z,\zeta)\hat{\phi}(\xi)\bar{\psi}(x)dxd\xi dzd\zeta
\end{aligned}$$

Similarly, $I_0^{\#} = (a\#b(x,D)\phi,\psi)$ is given by

$$I_0^{\#} = (2\pi)^{-2n}\int e^{i(\langle x,\xi\rangle-\langle x-z,\xi-\zeta\rangle)}a(x,\zeta)b(z,\xi)\hat{\phi}(\xi)\bar{\psi}(x)dxd\xi dzd\zeta$$

On the other hand, $I^* = (\phi, a(x,D)\phi) = (2\pi)^{-n}(\hat{\phi}, a(x,\hat{}D)\phi$ and $I^{\#} = (a(x,D)b(x,D)\phi,\psi)$ are given by

$$\begin{aligned}
I^* &= (2\pi)^{-2n}\int \hat{\phi}(\xi)(\int e^{i\langle z,\xi\rangle}(\int e^{-i\langle z,\zeta\rangle}\bar{a}(z,\zeta)(\int e^{i\langle x,\zeta\rangle}\bar{\psi}(x)dx)d\zeta)dz)d\xi\\
I^{\#} &= (2\pi)^{-2n}\int(\int e^{i\langle x,\zeta\rangle}a(x,\zeta)(\int e^{-i\langle z,\zeta\rangle}(\int e^{\langle z,\xi\rangle}b(z,\xi)\hat{\phi}(\xi)d\xi)dz)d\zeta)\bar{\psi}(x)dx
\end{aligned}$$

Thus it suffices to show that $I_0^* = I^*$ and $I_0^{\#} = I^{\#}$.

First, we show that $I_0^* = I^*$. Note that I_0^* is $\lim_{\epsilon\to 0} I_\epsilon^*$, where

$$I_\epsilon^* = (2\pi)^{-2n}\int \chi(\epsilon x)\chi(\epsilon\xi)\chi(\epsilon z)\chi(\epsilon\zeta)e^{i(\langle x,\xi\rangle-\langle x-z,\xi-\zeta\rangle)}\bar{a}(z,\zeta)\hat{\phi}(\xi)\bar{\psi}(x)dxd\xi dzd\zeta$$

where $\chi \in \mathcal{S}$ can be chosen so that $\chi = 1$ in B_1. Then we have $I^* - I^*_\epsilon = I^1_\epsilon + I^2_\epsilon + I^3_\epsilon$, where

$$I^1_\epsilon = (2\pi)^{-n} \int e^{i\langle z,\xi\rangle} \hat{\phi}(\xi)(1 - \chi(\epsilon\xi)\chi(\epsilon z))\bar{a}(z, D)\psi(z) d\xi dz$$

$$I^2_\epsilon = (2\pi)^{-2n} \int e^{i(\langle z,\xi\rangle - \langle z,\zeta\rangle)} \hat{\phi}(\xi)\bar{a}(z, \zeta)\chi(\epsilon\xi)\chi(\epsilon z)(1 - \chi(\epsilon\zeta))\overline{\hat{\psi}}(\zeta) d\xi dz d\zeta$$

$$I^3_\epsilon = (2\pi)^{-2n} \int e^{i(\langle x,\zeta\rangle + \langle z,\xi\rangle - \langle z,\zeta\rangle)} \hat{\phi}(\xi)\bar{a}(z, \zeta)\chi(\epsilon\xi)\chi(\epsilon z)\chi(\epsilon\zeta)(1 - \chi(\epsilon x))\bar{\psi}(x) d\xi dz d\zeta dx$$

The integral $I^1_\epsilon \to 0$ as $\epsilon \to 0$ by dominated convergence. The integrals I^2_ϵ and I^3_ϵ also go to 0 as $\epsilon \to 0$, by the following result, Lemma A.4. FIX THIS

(vi) Finally, we show that $I^\#_0 = I^\#$. (similar to above, add)

□

Lemma A.4. *Let $a(x, y) \in A^m(\mathbb{R}^n \times \mathbb{R}^p)$, ϕ be a real valued function, and $\chi, \psi, \upsilon \in \mathcal{S}$ with $\chi|_{B_1(0)} \equiv 1$. Then*

$$\lim_{\epsilon \to 0} \int e^{i\phi(x,y)} a(x, y)\upsilon(\epsilon x)(1 - \chi(\epsilon y))\bar{\psi}(y) dx dy = 0$$

Proof. Let I be the integral in the above limit. Setting $z = \epsilon x$ gives

$$I = \int e^{i\phi(z/\epsilon,y)} a(z/\epsilon, y)\upsilon(z)(1 - \chi(\epsilon y))\bar{\psi}(y)\epsilon^{-n} dz dy$$

By definition of $|||a|||_0$, we have that

$$\begin{aligned} |a(z/\epsilon, y)| &\leq |||a|||_0 (1 + |z/\epsilon|^2 + |y|^2)^{m/2} \\ &\leq |||a|||_0 \epsilon^{-m} (1 + |z|^2)^{m/2} (1 + |y|^2)^{m/2} \end{aligned}$$

When $y \in \operatorname{supp}(1 - \chi(\epsilon y))$, $|y| \geq 1/\epsilon$, and so

$$\begin{aligned} |\bar{\psi}(y)| &\leq |\psi|_{2(m+n+p)} \left(\frac{1 + |y|^2}{1 + p}\right)^{-m-n-p} \\ &\leq C_\psi \epsilon^{m+n+p} (1 + |y|^2)^{-\frac{m+n+p}{2}} \end{aligned}$$

when $y \in \operatorname{supp}(1 - \chi(\epsilon y))$. Thus,

$$|e^{i\phi(z/\epsilon,y)} a(z/\epsilon, y)\upsilon(z)(1-\chi(\epsilon y))\bar{\psi}(y)\epsilon^{-n}| \leq \epsilon^p |||a|||_0 C_\psi (1+|z|^2)^{m/2} |\upsilon(z)| (1+|y|^2)^{-\frac{n+p}{2}}$$

Integrating gives the desired result. □

Proposition A.5. (Leibniz's rule) Let $u \in \mathcal{D}'(\Upsilon)$, $a \in C^\infty(\Upsilon)$, and $P(\xi)$ be a polynomial in the n variables $\xi_1, \cdots, \xi_n$. If D_j denotes $-i\partial_{x_j}$ (only x derivatives, no ξ derivatives), and $P(D)$ is the differential operator obtained by replacing ξ_j with D_j, then

$$P(D)(au) = \sum_\alpha (D^\alpha a)(P^{(\alpha)}(D)u)/\alpha! \qquad (44)$$

where $P^{(\alpha)}(\xi) = \partial^{|\alpha|} P(\xi)/\partial\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} = i^{|\alpha|} D^\alpha P(\xi)$

Proof. If $\phi \in C_0^\infty(\Upsilon)$, recall the definitions

$$(D_k u)(\phi) := -u(D_k\phi)$$
$$(au)(\phi) := u(a\phi)$$

The basic product rule generalizes easily, as follows:

$$\begin{aligned}((D_k a)u)(\phi) + (a(D_k u))(\phi) &= u((D_k a)\phi) + (D_k u)(a\phi)\\ &= u((D_k a)\phi) - u(D_k(a\phi))\\ &= u((D_k a)\phi) - u((D_k a)\phi) - u(a(D_k\phi))\\ &= -u(a(D_k\phi))\\ &= (D_k(au))(\phi)\end{aligned} \tag{45}$$

Repeatedly applying (45), we see that

$$P(D)(au) = \sum_\alpha (D^\alpha a)Q_\alpha(D)u$$

for some polynomial Q_α in $D_1, \cdots, D_n$.

It remains to show that $Q_\alpha(D) = P^{(\alpha)}(D)/\alpha!$. Notice that $P(D)e^{i\langle x,\xi+\eta\rangle} = P(\xi+\eta)e^{i\langle x,\xi+\eta\rangle}$. Thus if we take for the moment $a(x) = e^{i\langle x,\xi\rangle}$ and $u(x) = e^{i\langle x,\eta\rangle}$, we have

$$\begin{aligned}P(\xi+\eta)e^{i\langle x,\xi+\eta\rangle} &= P(D)e^{i\langle x,\xi+\eta\rangle}\\ &= P(D)(e^{i\langle x,\xi\rangle}e^{i\langle x,\eta\rangle})\\ &= \sum_\alpha (D^\alpha e^{i\langle x,\xi\rangle})Q_\alpha(D)e^{i\langle x,\eta\rangle})\\ &= e^{i\langle x,\xi+\eta\rangle}\sum_\alpha \xi^\alpha Q_\alpha(\eta)\end{aligned}$$

and hence $P(\xi+\eta) = \sum_\alpha \xi^\alpha Q_\alpha(\eta)$. By Taylor's formula, $Q_\alpha(\eta) = P^{(\alpha)}(\eta)/\alpha!$, and we are finished. □

Appendix B. Index of notation

- If $u \in L^1$, we define the Fourier transform $\hat{u}$ of u by $\hat{u}(\xi) = \int e^{-i\langle x,\xi\rangle}u(x)dx$ (recall that the Fourier Transform extends continuously to functions $v \in L^2$). Thus we have the following formulas for $\phi \in \mathcal{S}$:
  - For $\alpha \in \mathbb{Z}_+^n$, we have $\widehat{D_x^\alpha\phi(\xi)} = \xi^\alpha\hat{\phi}(\xi)$ and $\widehat{x^\alpha\phi}(\xi) = (-D_\xi)^\alpha\hat{\phi}(\xi)$
  - For all $u \in L^1$, $(\hat{u},\phi) = (\check{u},\hat{\phi})$
  - (Inversion formula) $\hat{\hat{\phi}} = (2\pi)^n\check{\phi}$, i.e. $\phi(x) = (2\pi)^{-n}\int e^{i\langle x,\xi}\hat{\phi}(\xi)d\xi$
  - (Parseval's formula) For any $\psi \in \mathcal{S}$, $(\hat{\phi},\hat{\psi}) = (2\pi)^n(\phi,\psi)$
- H^s is the Sobolev space with exponent s, and
  $$\|u\|_s^2 = (2\pi)^{-n}\int(1+|\xi|^2)^s|\hat{u}(\xi)|^2 d\xi < \infty$$
  for $u \in H^s$.
- $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwarz space of rapidly decreasing functions
- $\mathcal{D}'(\Upsilon)$ denotes the space of distributions on Υ
- $\lambda^s(\xi) = (1+|\xi|^2)^{s/2}$
- If $u\bar{v} \in L^1$, then by definition $(u,v) = \int u(x)\bar{v}(x)dx$

- $D_j = -i\partial_j$ and $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$
- $A \subset\subset B$ means $\bar{A} \subset B$ and $\bar{A}$ compact.
- ${}^tP$ denotes the adjoint of P given by $\int (Pu)(v) = \int u({}^tPv)$ whenever u or v has compact support and both u and v are smooth. Notice this is *not* the transpose with respect to the inner product $(u, v) = \int u\bar{v}dx$.

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E-mail address: `nwr@math.washington.edu`