Classifying Smooth Cubic Surfaces up to Projective Linear Transformation

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Introduction

We would like to study the space of smooth cubic surfaces in $\mathbb{P}^3$ when each surface is considered only up to projective linear transformation. Brundu and Logar ([1], [2]) define an action of the automorphism group of the 27 lines of a smooth cubic on a certain space of cubic surfaces parametrized by $\mathbb{P}^4$ in such a way that the orbits of this action correspond bijectively to the orbits of the projective linear group PGL_4 acting on the space of all smooth cubic surfaces in the natural way. They prove several other results in their papers, but in this paper (the author's senior thesis at the University of Washington) we focus exclusively on presenting a reasonably self-contained and coherent exposition of this particular result. In doing so, we chose to slightly modify the action and ensuing proof, more aesthetically than substantially, in order to better reveal the intricate relation between combinatorics and geometry that underlies this problem. We would like to thank Professors Chuck Doran and Jim Morrow for much guidance and support.

The Space of Cubic Surfaces

Before proceeding, we need to define terms such as "the space of smooth cubic surfaces". Let W be a 4-dimensional vector-space over an algebraically closed field k of characteristic zero whose projectivization $\mathbb{P}(W) = \mathbb{P}^3$ is the ambient space in which the cubic surfaces we consider live. Choose a basis (x, y, z, w) for the dual vector-space W^*. Then an arbitrary cubic surface is given by the zero locus $V(F)$ of an element $F \in S^3W^* \subset k[x, y, z, w]$, where S^nV denotes the n^{th} symmetric power of a vector space V — which in this case simply means the set of degree three homogeneous polynomials. Let us write such a polynomial explicitly as

$$F(x, y, z, w) = a_1x^2y + a_2x^2z + a_3xy^2 + a_4xyz + a_5xyw + a_6xz^2 + a_7xzw + a_8y^2w + a_9yzw + a_{10}yw^2 + a_{11}x^3 + a_{12}x^w + a_{13}xw^2 + a_{14}y^3 + a_{15}y^2z + a_{16}yz^2 + a_{17}z^3 + a_{18}z^2w + a_{19}zw^2 + a_{20}w^3.$$

Then we can take $(a_1 : \ldots : a_{20})$ to be homogeneous coordinates for a projective space $\mathbb{P}^{19}$ whose points are in bijection with the cubic surfaces in $\mathbb{P}^3$

(and we often blur the distinction between these points p and the corresponding cubic surfaces S_p).

Definition: The projective space $\mathbb{P}^{19}$ constructed above is known as the *space of cubic surfaces*. If $\Sigma \subset \mathbb{P}^{19}$ is the closed subvariety of singular cubic surfaces (i.e. the set of points in $\mathbb{P}^{19}$ that correspond to singular cubic surfaces in $\mathbb{P}^3$), then $\mathbb{P}^{19} \setminus \Sigma$ is the *space of smooth cubic surfaces.*

We are interested in the effect of projective linear transformations (often called *projectivities*) on cubic surfaces. Given such a transformation $T : \mathbb{P}^3 \to \mathbb{P}^3$, $T \in \mathrm{PGL}_4$, it is clear that T sends (smooth) cubic surfaces to (smooth) cubic surfaces, i.e. it acts on the space of cubic surfaces $\mathbb{P}^{19}$ in such a way that the space of smooth cubic surfaces $\mathbb{P}^{19} \setminus \Sigma$ is stable under this action. We can understand this action explicitly as follows. Let $S_p = V(F)$, $F \in S^3W^* \subset k[x, y, z, w]$, be a cubic surface corresponding to the point $p \in \mathbb{P}^{19}$. Then $T \cdot p$ is the point in $\mathbb{P}^{19}$ corresponding to the cubic surface $V(T \cdot F)$, where $T \cdot F$ is defined to be the cubic form given by $F(T \cdot (x : y : z : w))$ (and this last action is the natural one of PGL_4 on $\mathbb{P}^3$).

Line Quintets

We now introduce a particular configuration of five projective lines in $\mathbb{P}^3$ that we call a *line quintet.* These objects are quite useful because they 'characterize' projective linear transformations in a way that will be made precise shortly.

Definition: An *line quintet* $(l_1, \ldots, l_5)$ is an ordered quintuple of lines in $\mathbb{P}^3$ such that:

- l_2 meets l_1, l_3, and l_5 in three distinct points.
- l_4 meets l_1 and l_3 in two distinct points.
- There are no other intersections among the five lines.

Notation:

1. Let us name these five points of intersection as follows: $A = l_1 \cap l_2, B = l_1 \cap l_4, C = l_3 \cap l_4, D = l_2 \cap l_3$, and $E = l_2 \cap l_5$.
2. If $T \in \mathrm{PGL}_4$ is a projectivity and $L = (l_1, \ldots, l_5)$ is a line quintet, then since T preserves incidence relations we get another line quintet $(T \cdot l_1, \ldots, T \cdot l_5)$ that we denote by $T \cdot L$.

The following two results are a crucial first step in classifying smooth cubic surfaces up to projectivity.

Proposition: Every smooth cubic surface in $\mathbb{P}^3$ contains a line quintet.

Proof: Indeed, we show in a later section that there are 25920 ways to choose a line quintet from among the 27 lines on a smooth cubic surface. □

Quintet Lemma: If $L = (l_1, \ldots, l_5)$ and $L' = (l'_1, \ldots, l'_5)$ are two line quintets, then there exists a unique projectivity $T \in \mathrm{PGL}_4$ such that $T \cdot L = L'$.

Proof: Unfortunately the author is still working through the details of this proof. $\square$

Corollary: If two projectivities agree on a line quintet, then they must be equal.

Proof: Suppose $T_1 \cdot L = T_2 \cdot L$. Then $T_2^{-1} T_1 \cdot L = L$. The Quintet Lemma says that there is a unique projectivity sending L to itself, which is clearly the identity transformation. Thus $T_2^{-1} T_1 = \mathtt{id}$, and so $T_1 = T_2$. $\square$

We may fix a particular line quintet L with nice (i.e. easy to work with) coordinates and consider the family of smooth cubic surfaces that contain L. The preceding results imply that every smooth cubic surface is projectively equivalent to a member of this family: an arbitrary smooth cubic surface S must contain a line quintet L_S, and by the Quintet Lemma there is a projectivity T sending L_S to L which clearly carries S to a smooth cubic surface containing L.

For the rest of this paper we fix the line quintet $\mathcal{L} = (l_1, \ldots, l_5)$ *given by* $l_1 = V(y, z), l_2 = V(x, y), l_3 = V(x, w), l_4 = V(x-w, y-z), l_5 = V(x-y, z+w)$.
The intersection points as defined above are
A = (0:0:0:1), B = (1:0:0:1), C = (0:1:1:0), D = (0:0:1:0), E = (0:0:1:-1).

If $S = V(F)$ is a cubic surface with coordinates $a_1, \ldots, a_{20}$, then the condition that S contains l_1, l_2, l_3 is equivalent to having $a_{11} = a_{12} = \ldots = a_{20} = 0$. Next, S contains $l_4 \iff F(x, y, y, x) = 0 \iff a_1 + a_2 + a_5 + a_7 + a_{10} = a_3 + a_4 + a_8 + a_9 = 0$ and it contains $l_5 \iff F(x, x, z, -z) = 0 \iff a_1 + a_3 = a_2 + a_4 - a_5 - a_8 = -a_7 - a_9 + a_{10} = 0$. The set of points satisfying these conditions is a linear subvariety of $\mathbb{P}^{19}$, and a little elementary algebra shows that it is 4-dimensional (i.e. these conditions are independent), so that the family of cubic surfaces containing $\mathcal{L}$ forms a projective space $\mathbb{P}^4$. If we take a_1, a_2, a_4, a_5, a_6 to be free parameters, then $a_3 = -a_1,\ a_7 = -a_1 - a_5 + a_4,\ a_8 = a_2 + a_4 - a_5,\ a_9 = a_1 - a_2 - 2a_4 + a_5,\ a_{10} = -a_2 - a_4$. Therefore, we have a natural injective linear map ϕ which sends a cubic surface in this family to the corresponding point in the space of cubic surfaces,

$$\phi : \mathbb{P}^4 \longrightarrow \mathbb{P}^{19}$$
$$(a : b : c : d : e) \longmapsto$$
$$(a : b : -a : c : d : e : -a - d + c : b + c - d : a - b - 2c + d : -b - c : 0 : \ldots : 0).$$

Of course, some of the cubic surfaces in this family (i.e. in the image of ϕ) are singular. Let us gather such surfaces into a set that we call $\Sigma_{\mathcal{L}}$,

$$\Sigma_{\mathcal{L}} := \phi^{-1}(\Sigma).$$

Remark: Later we will be able to give a precise description of $\Sigma_{\mathcal{L}}$.

This brings us one step closer to understanding the space of smooth cubic surfaces, as the following simple result illustrates.

Proposition: If $\psi : \mathrm{PGL}_4 \times (\mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}) \to \mathbb{P}^{19}$ is defined by $(T, q) \mapsto T \cdot \phi(q)$, then $\mathrm{im}(\psi) = \mathbb{P}^{19} \setminus \Sigma$.

Proof: If $q \in \mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$, then $\phi(q) \in \mathbb{P}^{19} \setminus \Sigma$, so $T \cdot \phi(q) \in \mathbb{P}^{19} \setminus \Sigma$ for all $T \in \mathrm{PGL}_4$ because the space of smooth cubic surfaces is stable under this action, hence $\mathrm{im}(\psi) \subseteq \mathbb{P}^{19} \setminus \Sigma$.

Conversely, if $S \in \mathbb{P}^{19} \setminus \Sigma$ (i.e. if S is a smooth cubic surface), then by an earlier comment S is projectively equivalent to a smooth cubic surface containing $\mathcal{L}$, say $T \cdot S \supset \mathcal{L}$ for some $T \in \mathrm{PGL}_4$. This means that $T \cdot S \in \phi(\mathbb{P}^4 \setminus \Sigma_{\mathcal{L}})$, say $T \cdot S = \phi(q)$ where $q \in \mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$. Then $S = T^{-1} \cdot \phi(q)$, so $S \in \mathrm{im}(\psi)$. $\square$

Therefore, we have a space $\mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$ that seems like a reasonable candidate for the space of orbits of smooth cubic surfaces under the action of PGL_4. However, this space is too large — in the sense that different points in the image of ψ correspond to the same orbit. To remedy this situation we need to carefully investigate the configuration of lines on a smooth cubic surface.

Finding the Twenty-Seven Lines

It is well-known that a smooth cubic surface contains 27 lines. In this section we describe an algorithmic method for determining these lines on a given smooth cubic surface $S \in \mathbb{P}^{19} \setminus \Sigma$. Along the way several concepts will be developed that are fundamental in determining how to reduce $\mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$ to a small enough space so that its points correspond bijectively to projectivity classes of smooth cubic surfaces.

Suppose l is a line contained in S. Since l lives in $\mathbb{P}^3$ we can represent it as the intersection of two projective planes, i.e. $l = V(f, g)$ for some linear forms $f, g \in S^1W^*$. It is convenient to work with the pencil of planes through l, which we denote by $\pi_l(p, q) := V(pf + qg)$. Indeed, we may see that any plane containing l is in this pencil by choosing coordinates so that $l = V(x, y)$ and then noting that an arbitrary plane $\pi = V(px + qy + rz + sw)$ contains l if and only if $r = s = 0$, or in other words, if and only if $\pi = \pi_l(p, q)$ for some $(p : q) \in \mathbb{P}^1$.

Since $S = V(F)$ has degree 3, when it is intersected with a plane the resulting object will be a plane cubic curve. However, if the plane intersecting S contains l, i.e. it is in the pencil $\pi_l(p, q)$, then this cubic curve will contain l (recall that by assumption $l \subset S$), so that the degree three homogeneous polynomial defining the curve will have a linear factor corresponding to l. Once this linear term is factored off, the resulting degree 2 polynomial defines a conic in the plane $\pi_l(p,q)$. Let us denote this conic by $C_l(p,q)$. We denote by $D_l(p,q)$ the determinant of this conic (more precisely, the determinant of the matrix associated to the quadratic form defining this conic). Thus $C_l(p,q)$ is singular if and only if $D_l(p,q)=0$ (see [3] if this is unfamiliar). Recall that a singular plane conic is the union of two (not necessarily distinct) lines.

Summarizing this construction we have,

Geometrically: $S \cap \pi_l(p,q) = l \cup C_l(p,q) = \begin{cases} l \cup l_1 \cup l_2, \text{ if } D_l(p,q)=0 \\ l \cup (\text{smooth conic}), \text{ if } D_l(p,q) \neq 0 \end{cases}$

Algebraically: $F|_{pf+qg=0} = \begin{cases} (\text{linear})(\text{linear})(\text{linear}), \text{ if } D_l(p,q)=0 \\ (\text{linear})(\text{quadratic}), \text{ if } D_l(p,q) \neq 0 \end{cases}$

Definition: Suppose l and l' are two non-skew lines on a smooth cubic surface and let π be the plane spanned by these two lines. Since π is in the pencil of planes through l, we know that $S \cap \pi$ is the union of l and a conic. However, since both l and l' are contained in this intersection, we see that the conic must contain l' and hence is the union of l' and another line. We denote this third line by $S_l^{l'}$.

Claim: Equivalently, we may say that given two non-skew lines $l, l' \subset S$, the line $S_l^{l'}$ is defined to be the unique line of S that meets both l and l'. Indeed, the line $S_l^{l'}$ defined above is coplanar with l and l' (namely all three lines lie in π), hence it must meet both these lines since two projective lines intersect if and only if they are coplanar. Conversely, if l'' is a line of S intersecting both l and l', say $l'' \cap l = P$ and $l'' \cap l' = Q$, then l'' is the line spanned by these two points, both of which lie in π, so l'' is contained in π as well and thus it must be the line $S_l^{l'}$. $\square$

For concreteness, let $l = V(x,y)$ and consider an arbitrary cubic surface containing l. Such cubic surfaces are precisely those with $a_{17}=a_{18}=a_{19}=a_{20}=0$ according to the coordinates specified earlier in this paper. Now $\pi_l(p,q) = V(px+qy)$, so if $p \neq 0$ we may restrict to this plane by making the substitution $x = -\frac{q}{p}y$. This yields the following:

$$F(-\tfrac{q}{p}y, y, z, w) =$$
$$\tfrac{y}{p^3}[(a_{14}p^3 - a_3p^2q + a_1pq^2 - a_{11}q^3)y^2 + (a_{15}p^3 - a_4p^2q + a_2pq^2)yz + (a_{16}p^3 - a_6p^2q)z^2 + (a_8p^3 - a_5p^2q + a_{12}pq^2)yw + (a_9p^3 - a_7p^2q)zw + (a_{10}p^3 - a_{13}p^2q)w^2]$$

The matrix associated to the above quadratic form is:

$$\begin{bmatrix} a_{14}p^3 - a_3p^2q + a_1pq^2 - a_{11}q^3 & a_{15}p^3 - a_4p^2q + a_2pq^2 & a_8p^3 - a_5p^2q + a_{12}pq^2 \\ a_{15}p^3 - a_4p^2q + a_2pq^2 & a_{16}p^3 - a_6p^2q & a_9p^3 - a_7p^2q \\ a_8p^3 - a_5p^2q + a_{12}pq^2 & a_9p^3 - a_7p^2q & a_{10}p^3 - a_{13}p^2q \end{bmatrix}$$

For each line l that we know to be contained in S, we will be able to use the determinant of this matrix to find many more lines of S. As will soon be shown, we can easily develop an algorithm for finding all the lines of S once we know any two non-skew lines of S.

Definition-Lemma: The determinant of the above matrix is a homogeneous polynomial in p and q of degree 9, but each term in it is divisible by p^4 so the corresponding conic splits into two linear factors if and only if $(p : q)$ is a root of the resulting degree 5 homogeneous polynomial which we denote by $\widetilde{D}_l(p, q)$. Thus given a line l contained in S, we may find all five roots of $\widetilde{D}_l(p, q)$, say $(p_i : q_i), \dots (p_5 : q_5)$, and the corresponding conics $C_l(p, q)$ will each split into two lines, both of which are contained in S, so that we will find up to 10 lines of S in this manner. We call these lines the *lines of S associated to l*.

Proposition: If $l_1, l_2 \subset S$ are two given non-skew lines of a smooth cubic surface S, then the 27 lines of S are exhausted by the lines associated to l_1, l_2, and $l_3 := S^{l_1}_{l_2}$.

Proof: Suppose l is an arbitrary line of S. Consider the plane π spanned by l_1 and l_2. By counting projective dimensions we know that l must intersect π, and since l is contained in S this point of intersection must be contained in $S \cap \pi = l_1 \cup l_2 \cup l_3$. Thus l must intersect some l_i for $i \in \{1, 2, 3\}$. But this means that l will show up as a factor of the conic corresponding to one of the roots of $\widetilde{D}_{l_i}(p, q)$, and hence l is associated to l_i. $\square$

Identifying the Singular Cubic Surfaces

With this machinery in place, we are almost ready to determine explicitly which cubic surfaces that contain the fixed line quintet $\mathcal{L}$ are singular.

Lemma: If S is an irreducible cubic surface containing a line l such that $\widetilde{D}_l(p, q)$ has 5 distinct roots, then S is smooth on $S \setminus l$.

Proof: We may assume that $l = V(x, y) \subset S = V(F)$, so that F is of the form in the above algorithm (i.e. $a_{17} = \ldots = a_{20} = 0$). Suppose that $P \notin l$ is a singular point. We can choose coordinates without loss of generality so that $P = (1 : 0 : 0 : 0)$. Because S is irreducible, the condition that P is a singular point means precisely that $\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = \frac{\partial F}{\partial z}(P) = \frac{\partial F}{\partial w}(P) = 0$. These partial derivatives are:

$\frac{\partial F}{\partial x} = 3a_{11}x^2 + 2a_1xy + 2a_2xz + 2a_{12}xw + a_3y^2 + a_4yz + a_5yw + a_7zw + a_{13}w^2$,

$\frac{\partial F}{\partial y} = a_1x^2 + 2a_3xy + a_4xz + a_5xw + 3a_{14}y^2 + 2a_{15}yz + 2a_8yt + a_{16}z^2 + a_9zw + a_{10}w^2$,

$\frac{\partial F}{\partial z} = a_2x^2 + a_4xy + a_7xt + a_{15}y^2 + 2a_{16}yz + a_9yt$,

$\frac{\partial F}{\partial w} = a_{12}x^2 + a_5xy + a_7xz + 2a_{13}xt + a_8y^2 + a_9yz + 2a_{10}yt.$

Thus vanishing at P means that $a_1 = a_2 = a_{11} = a_{12} = 0$, so

$$D_l(p,q) = \begin{vmatrix} a_{14}p^3 - a_3p^2q & a_{15}p^3 - a_4p^2q & a_8p^3 - a_5p^2q \\ a_{15}p^3 - a_4p^2q & a_{16}p^3 - a_6p^2q & a_9p^3 - a_7p^2q \\ a_8p^3 - a_5p^2q & a_9p^3 - a_7p^2q & a_{10}p^3 - a_{13}p^2q \end{vmatrix}$$

This is a rather messy polynomial, but with the help of a computer algebra system we see that the discriminant of $\widetilde{D}_l(p,q)$ with respect to both p and q vanishes, so that indeed there is a repeated root. $\square$

Proposition: An irreducible cubic surface S containing a pair of non-skew lines l_1, l_2 is smooth if and only if the polynomials $\widetilde{D}_{l_i}(p,q)$, $i = 1, 2$, have no repeated roots and S is smooth at the point $l_1 \cap l_2$.

Proof: Smoothness readily follows from this property by the preceding lemma because the hypothesis for the lemma is satisfied for both $\widetilde{D}_{l_1}(p,q)$ and $\widetilde{D}_{l_2}(p,q)$, hence S is smooth on $(S \backslash l_1) \cup (S \backslash l_2) = S \backslash (l_1 \cap l_2)$, and by assumption S is smooth at this remaining point $l_1 \cap l_2$.

Conversely, if S is smooth then obviously S is smooth at $l_1 \cap l_2$, so we just need to show that $\widetilde{D}_{l_i}(p,q)$ has no repeated roots for $i = 1, 2$. An easy way to see this with the tools we have developed thus far is to try to find the 27 lines that we know to exist[1] on S. We proved earlier that each of the 27 lines is associated to l_1, l_2, or $l_3 := S_{l_2}^{l_1}$. Suppose, by contradiction, that $\widetilde{D}_{l_1}(p,q)$ has at most 4 distinct roots. Then there will be at most $2 * 4 = 8$ lines associated to l_1, but l_2 and l_3 will be counted among them, so we only get at most 6 new lines. Then associated to l_2 we have at most the usual $2 * 5 = 10$ lines, although these include l_1 and l_3, so we only get at most 8 more new ones. Similarly there are no more than 8 new lines associated to l_3. Thus there are at most $3 + 6 + 8 + 8 = 25$ lines obtained in this fashion, contradicting the fact the we should have found all 27. $\square$

This allows us to determine which cubic surfaces that contain a given pair of lines are singular, hence in particular we can explicitly describe the set $\Sigma_{\mathcal{L}} \subset \mathbb{P}^4$ of singular cubic surfaces containing $\mathcal{L}$: if F is the equation of a general cubic surface in the family $\mathbb{P}^4$ of cubic surfaces containing a line quintet $(l_1, \ldots, l_5)$, and $A := l_1 \cap l_2$, then this set is given by

$$V(\tfrac{\partial F}{\partial x}(A), \tfrac{\partial F}{\partial y}(A), \tfrac{\partial F}{\partial z}(A), \tfrac{\partial F}{\partial w}(A)) \bigcup V(\text{Disc}(\widetilde{D}_{l_1}, p), \text{Disc}(\widetilde{D}_{l_1}, q)) \bigcup V(\text{Disc}(\widetilde{D}_{l_2}, p), \text{Disc}(\widetilde{D}_{l_2}, q)),$$

where $\text{Disc}(f(x_1, \ldots, x_n), x_i)$ denotes the discriminate of a multivariate polynomial with respect to the i^{th} variable (i.e. treating all $x_j, j \neq i$ as constants). In [1] this is explicitly calculated for $\mathcal{L}$ thereby producing a formula for $\Sigma_{\mathcal{L}}$.

[1] Brundu and Logar use computational methods related to the material developed in the preceding section to prove directly that every smooth cubic surface contains 27 lines. We decided to keep things cleaner by referring to the literature for this well-known fact.

The 27 Lines and Their Incidence Relations

Theoretically it is possible to use the above algorithm to find all 27 lines on a general smooth cubic surface and study the ways these lines intersect explicitly. However, we find it much cleaner to refer to more general theory (see [4]) in which each smooth cubic surface is obtained as the blow-up of 6 points of $\mathbb{P}^2$ in general position, so that the 27 lines have a natural interpretation in terms of exceptional divisors and strict transforms. This leads to the following notation for the lines:

- $E_1, \ldots, E_6$ correspond to the exceptional curves from blowing up 6 points in the plane.
- F_{ij} $(1 \leqslant i < j \leqslant 6)$ correspond to the strict transform of the $\binom{6}{2} = 15$ lines passing through each pair of these points.
- $G_1, \ldots, G_6$ correspond to the strict transform of the unique non-singular conic containing any 5 of these 6 points (note that this is well-defined because the 6 points are assumed to be in general position).

The incidence relations for this labeling of the lines is listed here in its entirety in Table 1 (two lines intersect if and only if their corresponding entry is 1).

Using the incidence table we may observe that every smooth cubic surface contains precisely $51840/2 = 25920$ line quintets: there are 27 choices for the line l_1, 10 for l_2, 8 for l_3, 4 for l_4, and 3 for l_5, thus $27 \cdot 10 \cdot 8 \cdot 4 \cdot 3 = 25920$ total ways to choose lines $(l_1, \ldots, l_5)$ that satisfy the axioms of a line quintet. The particular labeling we will use for the fixed line quintet $\mathcal{L}$ is $(E_1, G_4, E_2, G_3, E_3)$.

It is important to understand the automorphism group of this configuration of lines (i.e. the group of permutations in the symmetric group on 27 letters that preserve all incidence relations). A key fact is that any six mutually skew lines determine the rest of the lines. Hartshorne proves this directly, but once we know the structure of the lines, i.e. their incidence relations, we can verify this fact: having labeled any six mutually skew lines as E_i, we know that G_j is the unique line of S that meets all E_i for $i \neq j$, and F_{ij} is the unique line of S that meets E_i and E_j but is skew with all other E_k $(k \neq i, j)$. Thus each way to choose 6 mutually skew lines from among the lines on S uniquely determines an automorphism of the lines (by sending the E_i to these lines), and any automorphism must send 6 mutually skew lines to 6 mutually skew lines so the entire automorphism group is obtained in this way. By looking at the incidence table we see that there are 27 choices for E_1, 16 choices for E_2, 10 for E_3, 6 for E_4, 2 for E_5, and only 1 for E_6, so there are $27 \cdot 16 \cdot 10 \cdot 6 \cdot 2 \cdot 1 = 51840$ automorphisms. In fact, it turns out (see [5]) that this group is isomorphic to the Weyl group $\mathbb{E}_6$, but we do not actually need to work with this isomorphism.

In order to prove that there is an action of the automorphism group on the space of smooth cubic surfaces in such a way that the orbits of this action are

Table 1: The incidence matrix for the 27 lines on a smooth cubic surface.

	E_1	E_2	E_3	E_4	E_5	E_6	G_1	G_2	G_3	G_4	G_5	G_6	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{23}	F_{24}	F_{25}	F_{26}	F_{34}	F_{35}	F_{36}	F_{45}	F_{46}	F_{56}
E_1		0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
E_2	0		0	0	0	0	1	0	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0
E_3	0	0		0	0	0	1	1	0	1	1	1	0	1	0	0	0	1	0	0	0	1	1	1	0	0	0
E_4	0	0	0		0	0	1	1	1	0	1	1	0	0	1	0	0	0	1	0	0	1	0	0	1	1	0
E_5	0	0	0	0		0	1	1	1	1	0	1	0	0	0	1	0	0	0	1	0	0	1	0	1	0	1
E_6	0	0	0	0	0		1	1	1	1	1	0	0	0	0	0	1	0	0	0	1	0	0	1	0	1	1
G_1	0	1	1	1	1	1		0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
G_2	1	0	1	1	1	1	0		0	0	0	0	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0
G_3	1	1	0	1	1	1	0	0		0	0	0	0	1	0	0	0	1	0	0	0	1	1	1	0	0	0
G_4	1	1	1	0	1	1	0	0	0		0	0	0	0	1	0	0	0	1	0	0	1	0	0	1	1	0
G_5	1	1	1	1	0	1	0	0	0	0		0	0	0	0	1	0	0	0	1	0	0	1	0	1	0	1
G_6	1	1	1	1	1	0	0	0	0	0	0		0	0	0	0	1	0	0	0	1	0	0	1	0	1	1
F_{12}	1	1	0	0	0	0	1	1	0	0	0	0		0	0	0	0	0	0	0	0	1	1	1	1	1	1
F_{13}	1	0	1	0	0	0	1	0	1	0	0	0	0		0	0	0	0	1	1	1	0	0	0	1	1	1
F_{14}	1	0	0	1	0	0	1	0	0	1	0	0	0	0		0	0	1	0	1	1	0	1	1	0	0	1
F_{15}	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0		0	1	1	0	1	1	0	1	0	1	0
F_{16}	1	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0		1	1	1	0	1	1	0	1	0	0
F_{23}	0	1	1	0	0	0	0	1	1	0	0	0	0	0	1	1	1		0	0	0	0	0	0	1	1	1
F_{24}	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	1	0		0	0	0	1	1	0	0	1
F_{25}	0	1	0	0	1	0	0	1	0	0	1	0	0	1	1	0	1	0	0		0	1	0	1	0	1	0
F_{26}	0	1	0	0	0	1	0	1	0	0	0	1	0	1	1	1	0	0	0	0		1	1	0	1	0	0
F_{34}	0	0	1	1	0	0	0	0	1	1	0	0	1	0	0	1	1	0	0	1	1		0	0	0	0	1
F_{35}	0	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0		0	0	1	0
F_{36}	0	0	1	0	0	1	0	0	1	0	0	1	1	0	1	1	0	0	1	1	0	0	0		1	0	0
F_{45}	0	0	0	1	1	0	0	0	0	1	1	0	1	1	0	0	1	1	0	0	1	0	0	1		0	0
F_{46}	0	0	0	1	0	1	0	0	0	1	0	1	1	1	0	1	0	1	0	1	0	0	1	0	0		0
F_{56}	0	0	0	0	1	1	0	0	0	0	1	1	1	1	1	0	0	1	1	0	0	1	0	0	0	0	

in bijection with projectivity classes of smooth cubic surfaces, we need to work very closely with the lines on various cubic surfaces and maps between these lines that preserve incidence relations. It turns out to be quite convenient to interpret this problem in the context of graph theory since many of the ideas needed to study the lines on cubic surfaces for our purposes are standard ideas when dealing with graphs and maps between graphs. Let us now develop this connection.

A Graph-Theoretic Approach

We now restrict our attention to the category of graphs: the objects we consider are graphs and the morphisms between them are graph homomorphisms. Recall that a graph homomorphism

$$f : G = (V_G, E_G) \to H = (V_H, E_H)$$

is a set-map on the vertex sets $f : V_G \to V_H$ such that edges are preserved: if $p, q \in V_G$ are vertices connected by an edge $\overline{pq} \in E_G$ of G, then $\overline{f(p)f(q)} \in E_H$ is an edge of H. The only graph homomorphisms we will need in this paper are injective on the vertex sets, so that we do not need to worry about contracting two connected vertices down to one vertex with a loop. In fact, we can restrict our category to *simple* graphs without any problem. For the remainder of the paper we shall consider the graphs obtained from various sets of lines according to their incidence relations. That is, given a set of lines $L = \{l_i\}$ (which can be either actual lines embedded in $\mathbb{P}^3$ or abstract labels such as E_i, G_j, F_{kl}) we define a graph, also denoted by L, to have vertices indexed by the lines $\{l_i\}$ and edges $\overline{l_i l_j}$ if and only if $l_i \cap l_j \neq \emptyset$ for $i \neq j$. Let us now fix some notation for the objects that arise in the course of the proof.

Definition: Let $S \in \mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$ be a smooth cubic surface.

- $\mathbb{L} := \{E_1, \cdots, E_6, G_1, \cdots, G_6, F_{12}, \cdots, F_{56}\}$ is the graph given by the 27 abstract labels.
- $\mathbb{L}_S := \{l \subset S\}$ is the graph of the actual 27 lines on S.
- $L := \{E_1, G_4, E_2, G_3, E_3\}$ is the graph of the labels of a fixed line quintet.
- $L_S := \{l \in \mathbb{L}_S \mid l \in \mathcal{L}\}$ is the graph of the fixed line quintet used to define the space $\mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$.
- It is clear that there are natural inclusions $L \hookrightarrow \mathbb{L}$ and $L_S \hookrightarrow \mathbb{L}_S$. Additionally, if S' is another element of $\mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$, then by definition it contains the line quintet $\mathcal{L}$ as well. Thus there is a natural isomorphism $L_S \xrightarrow{\sim} L_{S'}$, which simply means that we may interpret the lines of $\mathcal{L}$ either as lines of S or as lines of S'.
- We proved earlier that for any non-skew lines l, l', the line $S_l^{l'}$ is the unique line of S that intersects both l and l'. Accordingly, we can define a similar

relation abstractly on the labels $\mathbb{L}$. Given two labels $l, l' \in \mathbb{L}$ such that $\overline{ll'}$ is an edge, we write $\mathbb{L}_l^{l'}$ for the unique label that is joined to both l and l' by an edge.

- $G := \mathrm{Aut}(\mathbb{L}) \cong \mathbb{E}_6$ is the automorphism group of the graph $\mathbb{L}$.

Given an arbitrary $S \in \mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$, it is clear that we can label the line quintet $\mathcal{L}$ according the labels in L — i.e. there is an isomorphism $L_S \xrightarrow{\sim} L$. In fact, since $\mathrm{Aut}(L) \cong \mathbb{Z}_2$ (as can easily be seen by looking at the graph of L), there are two such maps. Let us fix one and call it ϕ_S. We would like to know how many different ways there are to label all the lines of S once we have fixed this labeling for the 5 lines of the fixed line quintet. This is rephrased and answered precisely in the following proposition.

Proposition: Let $S \in \mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$, and let $\sigma \in G$ be the involution defined by permuting the indices 5 and 6, namely $E_5 \leftrightarrow E_6, G_5 \leftrightarrow G_6$, and $F_{i5} \leftrightarrow F_{i6}$ for $i \in \{1, \ldots, 4\}$. There are precisely two maps Φ_S that make the following diagram commute:

$$\begin{tikzcd} L_S \arrow[r, hook] \arrow[d, "\phi_S"'] & \mathbb{L}_S \arrow[d, "\Phi_S"'] \\ L \arrow[r, hook] & \mathbb{L} \end{tikzcd}$$

Moreover, they differ by composition with σ: if Φ_S is one such map, then $\sigma \circ \Phi_S$ is the other one.

Proof: A moment's thought reveals that the claimed result is equivalent to saying that the automorphism group $\widetilde{G}$ of the vertex-colored graph $\widetilde{\mathbb{L}}$, in which each label $l \in L$ is given a unique color and the remaining 22 labels are in a distinct sixth color class, is precisely the group $\{1, \sigma\}$. The first step in proving this reformulated version of the proposition is to show that fixing the 5 labels of L actually fixes many more labels.

Consider the set

$$X := \{l \in \mathbb{L} \mid g(l) = l \ \forall g \in G\}$$

We are initially only given that $L \subset X$. However, suppose l, l' are intersecting labels in X. Then we claim $\mathbb{L}_l^{l'} \in X$ as well. Indeed, it is clear that for any $g \in G$ we must have

$$g(\mathbb{L}_l^{l'}) = \mathbb{L}_{g(l)}^{g(l')} = \mathbb{L}_l^{l'}.$$

Using this relation inductively, we see that X contains E_i and G_i for $i = 1, \ldots, 4$, F_{ij} for $1 \leqslant i < j \leqslant 4$, and F_{56}. Indeed, $\mathbb{L}_{E_i}^{G_j} = F_{ij}$, so we quickly get $F_{13}, F_{14}, F_{23}, F_{24}, F_{34} \in X$. Next, $\mathbb{L}_{E_3}^{F_{i3}} = G_i$ for $i = 1, 2$, so $G_1, G_2 \in X$. Finally, $\mathbb{L}_{G_3}^{F_{34}} = E_4$, $\mathbb{L}_{E_1}^{G_2} = F_{12}$, and $\mathbb{L}_{F_{12}}^{F_{34}} = F_{56}$, so we get $E_4, F_{12}, F_{56} \in X$. It is clear from its definition that σ fixes each of the labels in L, hence each of the above labels as well, but it does not fix any other labels. Thus X is precisely equal to the set of labels described above. This proof will be complete once we show that σ is in fact the only automorphism with this property.

Suppose $g \in G$ satisfies $g(l) = l$ for each $l \in X$. We want to show $g \in \{1, \sigma\}$. Because E_5 intersects $G_1, \ldots, G_4$ and g preserves incidence relations, we must have that $g(E_5)$ intersects $g(G_1) = G_1, \ldots, g(G_4) = G_4$. By looking at the table of incidence relations, we see that this implies $g(E_5)$ is either E_5 or E_6. For the exact same reason we get that $g(E_6)$ is either E_5 or E_6, hence because g is an automorphism we see that g either permutes E_5 and E_6 or it fixes both of them. Similarly, g either permutes or fixes G_5 and G_6 because both of these labels intersect $g(E_1) = E_1, \ldots, g(E_4) = E_4$. Next, F_{15} and F_{16} both intersect E_1, G_1 and do not intersect $E_2, E_3, E_4, G_2, G_3, G_4$, all of which lines in X, so the incidence table allows us to see that g either fixes these two labels or permutes them. A completely analogous observation shows that g either permutes or fixes F_{i5} and F_{i6} for each of $i = 1, \ldots, 4$. Thus, it only remains to show that if g fixes any one of these pairs of labels, then it must fix all the other pairs as well. Indeed, if $g(E_5) = E_5$ then since E_5 intersects G_6 we get that $g(E_5) = E_5$ intersects $g(G_6)$, so we must have $g(G_6) = G_6$, and since $F_{i5} = \mathbb{L}_{E_5}^{G_i}$ we must have $g(F_{i5}) = g(\mathbb{L}_{E_5}^{G_i}) = \mathbb{L}_{g(E_5)}^{g(G_i)} = \mathbb{L}_{E_5}^{G_i} = F_{i5}$. Thus any automorphism that fixes each of the lines in L is either the identity or σ. $\square$

The Quintet Lemma tells us that projectivities are characterized by line quintets. As we see in the preceding proposition and corresponding proof, labeling the lines of a quintet only determines a labeling on all the lines of S up to an action of an order 2 permutation, but if we choose a label for a sixth line, say E_5, then there exists a unique way to extend the labeling $L \cup \{E_5\}$ to all the lines of S. This discrepancy of needing more information when working with the combinatorics of the lines than when working with their projective geometry gives rise to the heart of the theorem. In other words, it provides much of the subtlety, difficulty, and intrigue in finding the appropriate action of G on $\mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$.

Rather than working directly with cubic surfaces containing the labeled line quintet $\mathcal{L}$, we need to study cubic surfaces that have an addition line specified to play the role of E_5.

Definition: For each cubic surface $S \in \mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$, fix a labeling

$$\phi_S : L_S \xrightarrow{\sim} L$$

of its specified line quintet and an extension of this labeling

$$\Phi_S : \mathbb{L}_S \xrightarrow{\sim} \mathbb{L}.$$

We define a set $Z := \{(S, r) \mid \Phi_S(r) \in \{E_5, E_6\}\} \subset (\mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}) \times \mathrm{Gr}(2, 4)$, where $\mathrm{Gr}(k, n)$ denotes the Grassmannian variety of k-dimensional subspaces of an n-dimensional vector space (so in this case, projective lines in $\mathbb{P}^3$). Now for each $z = (S, r) \in Z$, let

$$\Phi_z := \begin{cases} \Phi_S \text{ if } \Phi_S(r) = E_5 \\ \sigma \circ \Phi_S \text{ if } \Phi_S(r) = E_6 \end{cases}$$

Thus Φ_z is the unique way to extend the labeling on L_S to all the lines of S in such a way that r is labeled E_5, where $z = (S, r)$.

Remark: By looking at the incidence table it is easy to see that $\Phi_S(r) \in \{E_5, E_6\}$ if and only if r intersects l_4 and l_5 but r does not intersect l_1, l_2 or l_3, where $L_S = (l_1, \ldots, l_5)$. This gives a way to define Z entirely from incidence relations.

By definition G is the automorphism group of the graph of labels $\mathbb{L}$. Now that we have a way to choose a well-defined labeling of the lines of S, we can extend this action of G so that it acts as automorphisms of $\mathbb{L}_S$ itself.

Definition: Given $z = (S, r) \in Z$ and $g \in G$, we define a graph homomorphism $g_z : \mathbb{L}_S \to \mathbb{L}_S$ by $g_z := \Phi_z^{-1} \circ g \circ \Phi_z$.

Proposition: For each $z = (S, r) \in Z$, the map $l \mapsto g_z(l)$ is an action of G on $\mathbb{L}_S$. Moreover, fixing such a z induces a group homomorphism

$$G = \mathrm{Aut}(\mathbb{L}) \to \mathrm{Aut}(\mathbb{L}_S)$$

given by $g \mapsto g_z$.

Proof: If $z = (S, r) \in Z, l \in \mathbb{L}_S$, and $g, h \in G$ then

$$g_z(h_z(l)) = \Phi_z^{-1}(g(\Phi_z(\Phi_z^{-1}(h(\Phi_z(l)))))) = \Phi_z^{-1}(g(h(\Phi_z(l)))) = (gh)_z(l).$$

It is clear that each g_z is a bijective graph map (i.e. a graph isomorphism) since it is the composition of such maps, so we certainly have a set map $\mathrm{Aut}(\mathbb{L}) \to \mathrm{Aut}(\mathbb{L}_S)$, but the previous line shows that in fact this is a group homomorphism. $\square$

We can now begin to develop the important bridge between the geometry we are interested in, namely projectivities, and the combinatorics we are developing, namely automorphisms and labelings of the lines on cubic surfaces.

Definition: Let $g \in G$ and $z = (S, r) \in Z$. We define $\widetilde{g_z} \in \mathrm{PGL}_4$ to be the unique projectivity given by the Quintet Lemma that satisfies

$$\widetilde{g_z} \cdot l = g_z(l)$$

for each $l \in L_S$. Note that $\widetilde{g_z}$ sends lines of S to lines of $\widetilde{g_z} \cdot S$ and it preserves incidence relations, so we can interpret $\widetilde{g_z}$ as a graph isomorphism $\mathbb{L}_S \xrightarrow{\sim} \mathbb{L}_{\widetilde{g_z} \cdot S}$.

As a useful reminder of how all the objects we have thus far defined relate, the reader may find it helpful to verify that the following diagram of graph homomorphisms commutes:

$$\begin{tikzcd}
g(L_S) \arrow[r, hook] & \mathbb{L}_{\widetilde{g_z}\cdot S} & \\
L_S \arrow[u, "g"] \arrow[r, hook] \arrow[d, "\phi_S"'] & \mathbb{L}_S \arrow[u, "\widetilde{g_z}"'] \arrow[r, "g_z"] \arrow[d, "\Phi_z"'] & \mathbb{L}_S \\
L \arrow[r, hook] & \mathbb{L} \arrow[r, "g"] & \mathbb{L} \arrow[u, "\Phi_z^{-1}"']
\end{tikzcd}$$

We can now define an action of G on the space Z which we will later descend via projection $\pi : (\mathbb{P}^4 \setminus \Sigma_{\mathcal{L}})\times \text{Gr}(2,4) \to \mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$ to an action on the space of smooth cubic surfaces containing $\mathcal{L}$.

Definition-Lemma: Given $g \in G$ and $z = (S, r) \in Z$, write

$$g(z) := (\widetilde{g_z}^{-1} \cdot S,\ \widetilde{g_z}^{-1} \cdot g_z(r))$$

This defines a *right* action of G on Z.

Proof: First, we need to verify that $g(z) \in Z$. Indeed, $\widetilde{g_z}^{-1} \cdot S$ is a smooth cubic surface and because S contains the line quintet $g_z(\mathcal{L})$ we know that $\widetilde{g_z}^{-1} \cdot S$ contains $\widetilde{g_z}^{-1} \cdot g_z(\mathcal{L}) = \mathcal{L}$, so we just need to see that $\widetilde{g_z}^{-1} \cdot g_z(r)$ intersects l_4, l_5 and is skew with l_1, l_2, l_3. This is quite straightforward: r must intersect l_4, because $(S, r) \in Z$, so $g_z(r)$ intersects $g_z(l_4)$, hence $\widetilde{g_z}^{-1} \cdot g_z(r)$ intersects $\widetilde{g_z}^{-1} \cdot g_z(l_4) = l_4$, and the rest of the necessary incidence relations follow in exactly the same manner.

Let us now verify that this is an action, i.e. that

$$(gh)(z) = h(g(z)),$$

or expanding out this expression according to the definition, we need to show

$$(\widetilde{gh_z}^{-1} \cdot S,\ \widetilde{gh_z}^{-1} \cdot (gh)_z(r)) = (\widetilde{h_{g(z)}}^{-1} \cdot \widetilde{g_z}^{-1} \cdot S,\ \widetilde{h_{g(z)}}^{-1} \cdot h_{g(z)}(\widetilde{g_z}^{-1} \cdot g_z(r))).$$

The first step is showing that

$$\widetilde{gh_z} = \widetilde{g_z} \circ \widetilde{h_{g(z)}}.$$

By the Quintet Lemma we just need to show that these two projectivities agree on each line of L_S. On the one hand we have

$$\widetilde{gh_z} \cdot l_1 = (gh)_z(l_1) = \Phi_z^{-1}(gh(\Phi_z(l_1))) = \Phi_z^{-1}(gh(E_1)).$$

On the other hand,

$$\widetilde{g}_z \cdot \widetilde{h_{g(z)}} \cdot l_1 = \widetilde{g}_z \cdot h_{g(z)}(l_1) = \widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(h(\Phi_{g(z)}(l_1))) = \widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(h(E_1)).$$

We claim that for each $l \in \mathbb{L}$ the following expression holds:

$$\widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(l) = \Phi_z^{-1}(g(l))$$

Assuming this for the moment, we quickly have that

$$\widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(h(E_1)) = \Phi_z^{-1}(g(h(E_1))) = \Phi_z^{-1}(gh(E_1)),$$

and the similar statement holds for the other lines $l_i \in L_S$, so indeed we will have that $\widetilde{gh}_z = \widetilde{g}_z \circ \widetilde{h_{g(z)}}$ once we verify the claim.

We prove the claim in stages. The first step is to verify it for labels $l \in L$. This is quite simple:

$$\widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(E_1) = \widetilde{g}_z \cdot l_1 = g_z(l_1) = \Phi_z^{-1}(g(\Phi_z(l_1))) = \Phi_z^{-1}(g(E_1)),$$

and similarly for E_2, E_3, G_3, and G_4. Next we show that it holds for E_5:

$$\widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(E_5) = \widetilde{g}_z \cdot (\widetilde{g}_z^{\,-1} \cdot g_z(r)) = g_z(r) = \Phi_z^{-1}(g(\Phi_z(r))) = \Phi_z^{-1}(g(E_5)).$$

Finally, using the fact that all labels of $\mathbb{L}$ can be obtained from $L \cup E_5$ using the $\mathbb{L}_l^{l'}$ construction, we will be done once we show that for any intersecting labels $l, l' \in \mathbb{L}$ such that the equality in the claim holds, i.e.

$$\widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(l) = \Phi_z^{-1}(g(l)) \text{ and } \widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(l') = \Phi_z^{-1}(g(l')),$$

then we also have

$$\widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(\mathbb{L}_l^{l'}) = \Phi_z^{-1}(g(\mathbb{L}_l^{l'})).$$

Indeed, given l, l' with this hypothesis, a little thought easily shows that

$$\widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(\mathbb{L}_l^{l'}) = \widetilde{g}_z \cdot (\widetilde{g}_z^{\,-1} \cdot S)_{\Phi^{-1}_{g(z)}(l)}^{\Phi^{-1}_{g(z)}(l')} = S_{\widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(l)}^{\widetilde{g}_z \cdot \Phi^{-1}_{g(z)}(l')} = S_{\Phi_z^{-1}(g(l))}^{\Phi_z^{-1}(g(l'))} = \Phi_z^{-1}(g(\mathbb{L}_l^{l'})).$$

This completes the proof of the claim and hence completes the proof of the lemma. $\square$

Definition-Lemma: Consider the natural projection

$$\pi : (\mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}) \;\times\; \mathrm{Gr}(2,4) \to \mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}.$$

Given a cubic surface $S \in \mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$, let $z \in \pi^{-1}(S)$ be an element in the fiber over S. The only other element in this fiber is $\sigma(z)$. More importantly, if we write

$$g \cdot S := \pi(g^{-1}(z)),$$

then this defines a *left* action of G on $\mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$.

Proof: If $z = (S, r)$ is in the fiber over S, we want to see that $\pi^{-1}(S) = \{z, \sigma(z)\}$. We already know that π has degree 2, so it enough to show that $\pi(\sigma(z)) = S$. This is equivalent to showing that $\widetilde{\sigma_z}$ is the identity transformation, and by the Quintet Lemma we only need to verify that it acts as the identity on L_S. Indeed,

$$\widetilde{\sigma_z} \cdot l_1 = \sigma_z(l_1) = \Phi_z^{-1}(\sigma(\Phi_z(l_1))) = \Phi_z^{-1}(\sigma(E_1)) = \Phi_z^{-1}(E_1) = l_1,$$

and similarly for the other elements of L_S. From this it immediately follows that the purported action is at least well-defined.

We now want to show that

$$(gh) \cdot S = g \cdot (h \cdot S).$$

Let $z \in \pi^{-1}(S)$ and $z' \in \pi^{-1}(\pi(h^{-1}(z)))$. By the preceding paragraph we know that $z' = h^{-1}(z)$ or $z' = \sigma(h^{-1}(z))$. Thus,

$$g \cdot (h \cdot S) = g \cdot \pi(h^{-1}(z)) = \pi(g^{-1}(z')) = \begin{cases} \pi(g^{-1}(h^{-1}(z))), \text{ or} \\ \pi(g^{-1}(\sigma(h^{-1}(z)))) \end{cases}$$

which in the first case is clearly $\pi((h^{-1}g^{-1})(z)) = \pi((gh)^{-1}(z)) = (gh) \cdot S$, since the action of G on Z is a right action. In the latter case it is enough to recall that $\widetilde{\sigma_x}$ is the identity transformation for any $x \in Z$, so the extra term of σ does not have any effect once we project down with π, and so the same equality holds. $\square$

Having verified the definition of this action, we will have completed the proof of the main theorem as soon as we show that the orbits of this action are in bijective correspondence with projectivity classes of smooth cubic surfaces.

Theorem: Suppose S, S' are two smooth cubic surfaces. There exists a projectivity $T \in \mathrm{PGL}_4$ satisfying $T \cdot S = S'$ if and only if there is an automorphism $g \in G$ satisfying $g \cdot S = S'$.

Proof: From what we have discussed previously in this paper regarding line quintets, it is enough to show this for cubic surfaces $S, S' \in \mathbb{P}^4 \setminus \Sigma_{\mathcal{L}}$. Suppose S and S' are in the same orbit, i.e. $S' = g^{-1} \cdot S = \pi(g(z))$, where $z = (S, r) \in \pi^{-1}(S)$. Then $g(z) = (\widetilde{g_z}^{-1} \cdot S,\ \widetilde{g_z}^{-1} \cdot g_z(r))$, so $S' = \pi(g(z)) = \widetilde{g_z}^{-1} \cdot S$ and hence S' is projectively equivalent to S.

Conversely, suppose that $S' = T \cdot S$ for some $T \in \mathrm{PGL}_4$. Let $z \in \pi^{-1}(S)$ and $z' \in \pi^{-1}(S')$. Consider the map $\mathbb{L} \to \mathbb{L}$ given by the composition:

$$\mathbb{L} \xrightarrow{\Phi_{z'}^{-1}} \mathbb{L}_{S'} \xrightarrow{T^{-1}} \mathbb{L}_S \xrightarrow{\Phi_z} \mathbb{L}$$

i.e. $l \mapsto \Phi_z(T^{-1} \cdot \Phi_{z'}^{-1}(l))$. This is certainly an automorphism because each map in the composition is a graph isomorphism, hence there exists $g \in G$ such that

$$g(l) = \Phi_z(T^{-1} \cdot \Phi_{z'}^{-1}(l)).$$

We claim that the projectivity g_z induces is precisely T^{-1}. By the Quintet Lemma it is enough to show that $\widetilde{g_z}$ sends each $l_i \in L_S$ to $T^{-1} \cdot l_i$. Indeed,

$$\widetilde{g_z} \cdot l_1 = g_z(l_1) = \Phi_z^{-1}(g(\Phi_z(l_1))) = \Phi_z^{-1}(\Phi_z(T^{-1} \cdot \Phi_{z'}^{-1}(\Phi_z(l_1)))) = T^{-1} \cdot \Phi_{z'}^{-1}(E_1) = T^{-1} \cdot l_1,$$

and similarly for the other $l_i \in L_S$. Thus $\widetilde{g_z} = T^{-1}$.

Using this allows us to see that

$$g(z) = (\widetilde{g_z}^{-1} \cdot S,\ \widetilde{g_z}^{-1} \cdot g_z(r)) = (T \cdot S,\ T \cdot g_z(r)),$$

and so $S' = T \cdot S = \pi(g(z)) = g^{-1} \cdot S$, as desired. $\square$

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