BIPARTITE RIGIDITY

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Abstract. We develop a bipartite rigidity theory for bipartite graphs parallel to the classical rigidity theory for general graphs, and define for two positive integers k, l the notions of (k, l)-rigid and (k, l)-stress free bipartite graphs. This theory coincides with the study of Babson–Novik's balanced shifting restricted to graphs. We establish bipartite analogs of the cone, contraction, deletion, and gluing lemmas, and apply these results to derive a bipartite analog of the rigidity criterion for planar graphs. Our result asserts that for a planar bipartite graph G its balanced shifting, G^b, does not contain $K_{3,3}$; equivalently, planar bipartite graphs are generically $(2, 2)$-stress free. We also discuss potential applications of this theory to Jockusch's cubical lower bound conjecture and to upper bound conjectures for embedded simplicial complexes.

1. Introduction

1.1. **Three basic properties of planar graphs.** We start with three important results on planar graphs and a motivating conjecture in a higher dimension:

Proposition 1.1 (Euler, Descartes). *A simple planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges.*

Proposition 1.2 (Wagner, Kuratowski (easy part)). *A planar graph does not contain K_5 and $K_{3,3}$ as minors.*

The first result (that can be traced back to Descartes) is a simple consequence of Euler's theorem. The second result is the easy part of Wagner's characterization of planar graphs [41] which asserts that not having K_5 and $K_{3,3}$ as minors characterizes planarity.

We will now state a third fundamental result on planar graphs. This requires some definitions. An embedding of a graph G into $\mathbb{R}^d$ is a map assigning a vector $\phi(v) \in \mathbb{R}^d$ to every vertex v. The embedding is *stress-free* if there is no way to assign weights w_{uv} to edges, so that not all weights are equal to zero and every vertex is "in equilibrium":

$$\sum_{v \,:\, uv \in E(G)} w_{uv}(\phi(u) - \phi(v)) = 0 \quad \text{for all } u. \tag{1}$$

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The embedding is *infinitesimally rigid* if every assignment of velocity vectors $V(u) \in \mathbb{R}^d$ to vertices of G that satisfies

$$\langle V(v) - V(u), \phi(v) - \phi(u) \rangle = 0 \tag{2}$$

for every $uv \in E(G)$, must satisfy relation (2) for *every pair* of vertices.[1]

Proposition 1.3 (Gluck, Dehn, Alexandrov, Cauchy)**.** *A generic embedding of a simple planar graph in $\mathbb{R}^3$ is stress free. A generic embedding of a maximal simple planar graph in $\mathbb{R}^3$ is also infinitesimally rigid.*

This result of Gluck [13] is closely related to Cauchy's rigidity theorem for polytopes of dimension three, and is easily derived from its infinitesimal counterpart by Dehn and Alexandrov. We refer our readers to [10, 32] for an exposition and further references. The above three results on planar graphs also have interesting inter-connections.

In this paper we consider extensions of these three results to bipartite graphs. The extensions to bipartite graphs are of interest on their own and they are also offered as an approach toward hard higher-dimensional generalizations such as the following conjecture that in a slightly different form was raised as a question by Grünbaum [15, Section 3.7]. For a simplicial complex K let $f_i(K)$ denote the number of i-dimensional faces of K.

Conjecture 1.4. *There is an absolute constant C such that a 2-dimensional simplicial complex K embedded in $\mathbb{R}^4$ satisfies*

$$f_2(K) \le C f_1(K).$$

1.2. **Bipartite rigidity and a bipartite analog of Gluck's theorem.** We develop a bipartite analog for the rigidity theory of graphs.

A (k, l)-*embedding* of a bipartite graph $G = (A \uplus B, E)$ is a map $\phi : A \uplus B \to \mathbb{R}^k \times \mathbb{R}^l$ that assigns to every $a \in A$ a vector $\phi(a) \in \mathbb{R}^k \times (0)$, and to every $b \in B$ a vector $\phi(b) \in (0) \times \mathbb{R}^l$. A (k, l)-embedding ϕ of a bipartite graph $G = (A \uplus B, E)$ is (k, l)-*stress free* if there is no way to assign weights w_{ab} to edges so that not all weights are equal to zero and every vertex u satisfies:

$$\sum_{v\,:\,uv \in E} w_{uv}\phi(v) = 0. \tag{3}$$

A (k, l)-embedding ϕ of a bipartite graph G is (k, l)-*rigid* if every assignment of velocity vectors $V(a) \in (0) \times \mathbb{R}^l$ for $a \in A$ and $V(b) \in \mathbb{R}^k \times (0)$ for $b \in B$ that satisfies

$$\langle V(a), \phi(b) \rangle + \langle V(b), \phi(a) \rangle = 0 \tag{4}$$

for all $ab \in E$, must satisfy equation (4) for all $ab \in A \times B$.

It is worth mentioning (see Remark 3.5 and Theorem 5.4) that (k, k)-rigidity is equivalent to Kalai's hyperconnectivity [19] (restricted to the bipartite case), while $(k, 1)$-stress

[1] Relation (2) asserts that the velocities respect (infinitesimally) the distance along an embedded edge. If these relations apply to all pairs of vertices the velocities necessarily come from a rigid motion of the entire space.

freeness can be traced to the work of Whiteley [44]. In addition, (k, k)-rigidity is also equivalent to the Singer and Cucuringu's notion of *rectangular local completability in dimension* k as defined in [36, Section 4] in relation to the problem of completing a low-rank matrix from a subset of its entries.

In Section 4 we prove the following bipartite analog of Gluck's theorem.

Theorem 1.5. *A generic* $(2, 2)$*-embedding of a simple planar bipartite graph is* $(2, 2)$*-stress free. A generic* $(2, 2)$*-embedding of a maximal simple planar bipartite graph is also* $(2, 2)$*-rigid.*

Our theory of bipartite rigidity relies on the notion of "balanced shifting" that we sketch below.

1.3. **Shifting, balanced shifting, and bipartite graphs.** Algebraic shifting is an operation introduced by Kalai [18, 21, 22] that replaces a simplicial complex K with a "shifted" simplicial complex $\Delta(K)$. There are two versions of algebraic shifting: the symmetric one and the exterior one; we write $\Delta(K) = K^s$ in the former case, and $\Delta(K) = K^e$ in the latter case. For graphs, shifting is closely related to infinitesimal rigidity. The shifting operation preserves various properties of the complex, and, in particular, the numbers of faces of every dimension. In dimension one, shifted graphs are known as threshold graphs: the vertices numbered $\{1, 2, \ldots, n\}$ are assigned nonnegative weights $w_1 > w_2 > w_3 > \cdots > w_n$, and edges correspond to pairs of vertices with sum of weights above a certain threshold.[2]

The following result (see [22, 31]) expresses Gluck's theorem (Proposition 1.3) in terms of symmetric shifting, and clearly implies Euler's inequality of Proposition 1.1:

Proposition 1.6. *If* G *is a planar graph then the symmetric algebraic shifting of* G*,* G^s*, does not contain* K_5 *as a subgraph. Equivalently,* G^s *does not contain the edge* $\{4, 5\}$*.*[3] *More generally, the same statement holds for every graph* G *that does not contain* K_5 *as a minor.*

One drawback of this result is that G^s may contain $K_{3,3}$ and hence the planarity property itself is lost under shifting.

Similarly, the following conjecture implies Conjecture 1.4 with the sharp constant $C = 4$:

Conjecture 1.7. *If* K *is a 2-dimensional simplicial complex embeddable in* $\mathbb{R}^4$ *then* K^s *does not contain the* 2*-face* $\{5, 6, 7\}$*.*

We now move from graphs to *bipartite* graphs and, more generally, in higher dimensions from simplicial complexes to *balanced* simplicial complexes. A d-dimensional simplicial complex is called *balanced* if its vertices are colored with $d+1$ colors in such a way that every edge is bicolored; thus the $d+1$ colors for the vertices of every d-simplex

[2]In higher dimensions the class of shifted complexes is much reacher than the class of threshold complexes.

[3]A shifted graph contains K_5 as a subgraph if and only if it contains it as a minor.

are all different. A balanced 1-dimensional complex is simply a bipartite graph. The study of enumerative and algebraic properties of balanced complexes was initiated by Stanley [38].

Babson and Novik [4] defined a notion of balanced shifting, and associated with every balanced simplicial complex K a balanced-shifted complex K^b. We recall this operation in Section 2 (see also Section 7) mainly concentrating on the case of graphs. In Section 3, we show that in this case the properties of the balanced-shifted bipartite graphs are described in terms of "bipartite rigidity" as defined in Section 1.2. Specifically, we establish bipartite analogs of the cone, contraction, deletion, and gluing lemmas; then in Section 4 we use these results to prove Theorem 1.5 expressed in terms of balanced shifting as follows:

Theorem 1.8. *For a bipartite planar graph G, G^b does not contain $K_{3,3}$.*

A balanced-shifted bipartite graph without $K_{3,3}$ is planar, and therefore Theorem 1.8 implies that, in contrast with the case of symmetric shifting, the planarity property *is preserved* under balanced shifting. In other words, Theorem 1.8 settles the $d = 1$ case of the following conjecture.

Conjecture 1.9. *Balanced shifting for d-dimensional balanced complexes preserves embeddability in $\mathbb{R}^{2d}$.*

In Section 8 we discuss several variations as well as a more detailed version of this conjecture. It is also worth remarking that the $d = 2$ case of Conjecture 1.9 implies Conjecture 1.4 with $C = 9$, see Section 8 for more details.

We also discuss (see Section 6) a rigidity approach and some partial results regarding the following conjecture of Jockusch [16]:

Conjecture 1.10 (Jockusch)**.** *If K is a cubical polytope of dimension $d \geq 3$, with V vertices and E edges then $E \geq \frac{d+1}{2}V - 2^{d-1}$.*

The structure of the rest of the paper is as follows. In Section 2 we discuss basics of graphs as well as recall how to compute the balanced shifting, G^b, of a bipartite graph G. In Section 3, we define the notions of bipartite rigidity and stress freeness, and establish bipartite analogs of the cone, deletion, contraction, and gluing lemmas. In Section 4, we use these lemmas to prove Theorem 1.8; we also discuss there balanced shifting of linklessly embeddable graphs. In Section 5 we consider bipartite analogs of Laman's theorem; this includes analyzing balanced shifting of bipartite trees and outerplanar graphs. Section 6 is devoted to graphs of cubical polytopes (and, more specifically, to Jockusch's conjecture) as well as to graphs of polytopes that are dual to balanced simplicial polytopes. In the remaining sections we turn to higher-dimensional simplicial complexes: in Section 7 we recall basics of simplicial complexes, and in Section 8 we discusses several problems and partial results related to Conjecture 1.9.

Our rigidity theory of bipartite graphs has also led to purely graph-theoretic questions regarding bipartite graphs that we study separately in a joint work with Chudnovsky and Seymour [8]. Understanding the higher-dimensional analogs of graph minors for general complexes has been quite fruitful in establishing partial results on Conjecture 1.4, see [30, 42]. Finding such a notion for balanced complexes may be equally useful, and bipartite graphs are a good place to start. In [8] we initiate this program by defining a notion of bipartite minors and proving a bipartite analog of Wagner's theorem: a bipartite graph is planar if and only if it does not have $K_{3,3}$ as a bipartite minor.

2. Preliminaries on bipartite graphs and balanced shifting

All the graphs considered in this paper are simple graphs. A graph with the vertex set V and the edge set E is denoted by $G = (V, E)$. A graph is *bipartite* if there exists a bipartition of the vertex set V of G, $V = A \uplus B$, in such a way that no two vertices from the same part form an edge. When discussing bipartite graphs, we fix such a bipartition and write $G = (A \uplus B, E)$; we refer to A and B as *parts* or *sides* of G.

If $G = (V, E)$ is a graph and v is a vertex of G, then $G - v$ denotes the induced subgraph of G on the vertex set $V - \{v\}$. If $G = (V = A \uplus B, E)$ is a bipartite graph and u, v are two vertices from the *same* part, then the *contraction* of u with v is the graph G' on the vertex set $V - \{u\}$ obtained from G by identifying u with v and deleting the extra copy from each double edge that was created. Observe that G' is also bipartite.

We usually identify A and B with the ordered sets $\{1 < 2 < \ldots < n\} := [n]$ and $\{1' < 2' < ... < m'\} := [m']$, respectively. For brevity, we denote the edge connecting vertices i and j' by ij' (instead of $\{i, j'\}$). Define $\mathcal{E} = \mathcal{E}_{n,m} := \{ij' : i \in [n], j' \in [m']\}$ to be the edge set of the complete bipartite graph $K_{A,B} = K_{n,m}$ on $V = A \uplus B$. We also consider a total order, $<$, on V that extends the given orders on A and B, and the induced lexicographic order, $<_{\text{lex}}$, on $\mathcal{E}$.

Given a bipartite graph G and such an order $<$ on V, one can compute the *balanced shifting* of G, $G^b = G^{b,<}$. This notion was introduced in [4] for a much more general class of simplicial complexes (and was called "colored shifting" there). For the sake of completeness and to establish notation, we briefly recall here the relevant definitions.

Let $G = (A \uplus B, E)$ be a bipartite graph, and let $\mathbb{R}$ be the field of real numbers (although all the theory we develop here works over any infinite field). Consider two sets of variables: $\{x_1, \ldots, x_n\}$ (one x for each element of A) and $\{y_1, \ldots, y_m\}$ (one y for each element of B). Let S be a polynomial ring over $\mathbb{R}$ in the x's and y's, let I_G be the Stanley-Reisner ideal of G:

$$I_G = \langle \{x_i x_j : 1 \le i < j \le n\} \cup \{y_i y_j : 1 \le i < j \le m\} \cup \{x_i y_j : ij' \notin E\}\rangle,$$

and let $\mathbb{R}[G] := S/I_G$ be the Stanley-Reisner ring of G. Setting $\deg x_i := (1, 0)$ for all $i \in [n]$ and $\deg y_j := (0, 1)$ for all $j \in [m]$ makes $R[G]$ into a $\mathbb{Z}^2$-graded ring. For $(p, q) \in \mathbb{Z}^2$ we denote by $\mathbb{R}[G]_{(p,q)}$ the (p, q)-th homogeneous component of $R[G]$.

Let $\Theta \in \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_m(\mathbb{R})$ be a matrix whose entries $\theta_{ij}, \theta_{s't'} \in \mathbb{R}$ for $i, j \in [n]$ and $s', t' \in [m']$ are "generic" (for instance, algebraically independent over $\mathbb{Q}$ is more than enough). Set $\theta_i := \sum_{j=1}^{n} \theta_{ij} x_j$ (for $1 \le i \le n$) and $\theta_{s'} := \sum_{t=1}^{m} \theta_{s't'} y_t$ (for $1' \le s' \le m'$). The matrix Θ acts on the set of linear forms of $\mathbb{R}[G]$ by $\Theta x_i := \theta_i$ and $\Theta y_s := \theta_{s'}$, and this action can be extended uniquely to a ($\mathbb{Z}^2$-grading preserving) ring automorphism of $\mathbb{R}[G]$ that we also denote by Θ.

Given a total order $<$ on $A \uplus B$, define $G^b = G^{b,<}$ — the *balanced shifting* of G — as the bipartite graph whose vertex set is $A \uplus B$ and whose edge set, E^b, is given by

$$\left\{ ij' \in \mathcal{E} \ : \ \theta_i \theta_{j'} \notin \mathrm{Span}\{\theta_p \theta_{q'} : pq' <_{\mathrm{lex}} ij'\} \subset \mathbb{R}[G]_{(1,1)} \right\}.$$

In other words, the edge set of G^b is determined by the "greedy" lexicographic basis of the vector space $\mathbb{R}[G]_{(1,1)}$ chosen from the monomials written in θ's. For instance, $(K_{n,m})^{b,<} = K_{n,m}$ for any order $<$.

The following two properties of the balanced shifting from [4] will be handy:

Lemma 2.1. *For a bipartite graph $G = (A \uplus B, E)$ and any order $<$ on $A \uplus B$ that extends the natural orders on A and B, we have*

- $|E^b| = |E|$, *and*
- G^b *is* balanced-shifted*: if $ij' \in E^b$, $1 \leq p \leq i$, and $1' \leq q' \leq j'$, then $pq' \in E^b$.*

We finish this section with the following definition and observation that will be useful in the rest of the paper.

Definition 2.2. Given a pair of two fixed integers $k \leq n$ and $l \leq m$, we say that a total order on $V = A \uplus B$ is (k,l)-*admissible* if (i) it extends the natural orders on A and B, and (ii) the set $[k] \cup [l']$ forms an initial segment of V w.r.t. $<$.

Lemma 2.3. *Let G be a bipartite graph, let $<$ be a (k,l)-admissible order, and let $\Phi = \Phi_G^{(k,l)} : \left(\mathbb{R}[G]_{(1,0)}\right)^l \oplus \left(\mathbb{R}[G]_{(0,1)}\right)^k \longrightarrow \mathbb{R}[G]_{(1,1)}$ be the following linear map:*

$$(f_1, f_2, \ldots, f_l, g_1, \ldots, g_k) \mapsto \sum_{i=1}^{l} \theta_{i'} f_i + \sum_{j=1}^{k} \theta_j g_j.$$

Then

1. *all elements of $\mathcal{E}^{kl} := \{ij' \in \mathcal{E} : i \leq k \text{ or } j' \leq l'\}$ are edges of $G^{b,<}$ if and only if the dimension of the image of Φ equals $ln + km - kl$;*
2. *the pair $(k+1)(l+1)'$ is not an edge of $G^{b,<}$ if and only if Φ is surjective.*

Proof. Since $\{\theta_1, \ldots, \theta_n\}$ is a basis of $R[G]_{(1,0)}$ and $\{\theta_{1'}, \ldots, \theta_{m'}\}$ is a basis of $R[G]_{(0,1)}$, the set $\{\theta_i \theta_{j'} : ij' \in \mathcal{E}^{kl}\}$ is a spanning set of the image of Φ. On the other hand, by (k,l)-admissibility of the order $<$, $\mathcal{E}^{kl}$ is an initial segment of $\mathcal{E}$ w.r.t $<_{\mathrm{lex}}$. Hence, by the definition of $G^{b,<}$, $\mathcal{E}^{kl} \subseteq E^b$ if and only if $\{\theta_i \theta_{j'} : ij' \in \mathcal{E}^{kl}\}$ is a linearly independent subset of $\mathbb{R}[G]_{(1,1)}$. Therefore, $\mathcal{E}^{kl} \subseteq E^b$ if and only if $\{\theta_i \theta_{j'} : ij' \in \mathcal{E}^{kl}\}$ is a basis of the image of Φ. Part 1 follows.

The reasoning for Part 2 is similar: since G^b is balanced-shifted, the pair $(k+1)(l+1)'$ is not an edge of G^b if and only if $E^b \subseteq \mathcal{E}^{kl}$. Further, the fact that $\mathcal{E}^{kl}$ is an initial segment of $\mathcal{E}$ w.r.t $<_{\mathrm{lex}}$ yields that $E^b \subseteq \mathcal{E}^{kl}$ if and only if $\{\theta_i \theta_{j'} : ij' \in \mathcal{E}^{kl}\}$ is a spanning set of $R[G]_{(1,1)}$, which implies Part 2. $\square$

It is worth noting that since G^b is balanced-shifted, the edge $(k+1)(l+1)'$ is not an edge of G^b if and only if G^b does not contain $K_{k+1,l+1}$ as a subgraph.

3. (k,l)-RIGIDITY

The goal of this section is to develop a rigidity theory for bipartite graphs, paralleling the one for general graphs [1, 2, 43, 45]. We recall from [22, Section 2.7] and [25], as discussed in detail in [29, Section 3.2], that a (non-bipartite) graph G on the vertex set $[n]$ is *generically d-stress free* if and only if the pair $(d+1)(d+2)$ is *not* an edge of the symmetric shifting of G, G^s, and that G is *generically d-rigid* if and only if the pair dn *is* an edge of G^s. Motivated by these results, we make the following definition. We use the same notation as in the previous section.

Definition 3.1. Let $G = (A \uplus B, E)$ be a bipartite graph, let $k \le n$ and $l \le m$ be two fixed integers, and let $<$ be a (k,l)-admissible order on $A \uplus B$. We call G (generically) (k,l)-*stress free* if the pair $(k+1)(l+1)'$ is not an edge of $G^{b,<}$. We say that G is (generically) (k,l)-*rigid* if all pairs $ij' \in \mathcal{E}$ such that $i \le k$ or $j' \le l'$ are edges of $G^{b,<}$.

It follows from Lemma 2.3 that being (k,l)-rigid ((k,l)-stress free, respectively) does not depend on a particular choice of a (k,l)-admissible order $<$. In fact, writing the matrix of the map $\Phi_G^{(k,l)}$ from Lemma 2.3 with respect to the basis $(x_iy_j : ij' \in E)$ of $\mathbb{R}[G]_{(1,1)}$ and the basis

$$(l \text{ copies of } x_1, l \text{ copies of } x_2, \dots, l \text{ copies of } x_n, k \text{ copies of } y_1, \dots, k \text{ copies of } y_m)$$

of $\left(\mathbb{R}[G]_{(1,0)}\right)^l \oplus \left(\mathbb{R}[G]_{(0,1)}\right)^k$ yields the following definition and proposition.

Definition 3.2. Let $G = (A \uplus B, E)$ be a bipartite graph and let $\Theta \in \mathrm{GL}_n(\mathbb{R}) \times GL_m(\mathbb{R})$ be a block-generic matrix as in the previous section. Let $R^{(k,l)}(G)$ be an $|E| \times (l|A| + k|B|)$ matrix whose rows are labeled by the edges of G, whose columns occur in blocks of size l for each vertex in A and blocks of size k for each vertex in B, and whose block corresponding to $v \in V$ and $ab' \in E$ is given by

$$\begin{cases} (\theta_{i'b'} : \ 1 \le i \le l) & \text{if } v = a, \\ (\theta_{ia} : \ 1 \le i \le k) & \text{if } v = b, \\ 0 & \text{if } v \notin \{a, b\}. \end{cases}$$

The matrix $R^{(k,l)}(G)$ is called the *bipartite (k,l)-rigidity matrix of G*.

Proposition 3.3. *Let $G = (A \uplus B, E)$ be a bipartite graph. Then G is (k,l)-stress free if and only if the rows of $R^{(k,l)}(G)$ are linearly independent, and G is (k,l)-rigid if and only if* $\mathrm{rank}(R^{(k,l)}(G)) = l|A| + k|B| - kl$.

Several remarks are in order. For a matrix M, let $\mathrm{row}(M)$ denote the span of the rows of M.

Remark 3.4. Observe that for any bipartite graph $G = (A \uplus B, E)$, $\mathrm{row}(R^{(k,l)}(G)) \subseteq \mathrm{row}(R^{(k,l)}(K_{A,B}))$. On the other hand, Proposition 3.3 implies that G is (k,l)-rigid if and only if $\mathrm{rank}(R^{(k,l)}(G)) = l|A| + k|B| - kl$. Since $K^b_{A,B} = K_{A,B}$, the graph $K_{A,B}$ is (k,l)-rigid, and so

$$\mathrm{rank}(R^{(k,l)}(K_{A,B})) = l|A| + k|B| - kl.$$

Therefore, G is (k,l)-rigid if and only if $\mathrm{row}(R^{(k,l)}(G)) = \mathrm{row}(R^{(k,l)}(K_{A,B}))$.

In particular, it follows that as for classical combinatorial rigidity theory, there is a matroid underlying bipartite rigidity theory — namely the matroid represented by the rows of the (k,l)-rigidity matrix of $K_{A,B}$. A (k,l)-rigid graph is one whose edges are a spanning set for this matroid; a (k,l)-stress free graph is one whose edges are independent; and a (k,l)-rigid and stress free graph is a basis.

It is also worth remarking that in the case of $k = l$, our rigidity matrix coincides with the completion matrix defined in [36, eq. (4.2)]. As a result, a bipartite graph is (k,k)-rigid in our sense if and only if it is a rectangular graph that is locally completable in dimension k in the sense of [36]. (Given a matrix some of whose entries are known, the associated "rectangular" graph is the bipartite graph whose vertices correspond to rows and columns of the matrix and whose edges correspond to the known matrix entries.)

In addition, in the case of $k = l$, our rigidity matrix is related to Kalai's hyperconnectivity matrix [19], $H^{k,<'}(G)$, computed w.r.t. the entries of Θ and a total order $<'$ on the vertices. The rank of $H^{k,<'}(G)$ is independent of the total order chosen (this follows from the fact that the result of exterior algebraic shifting is independent of the labeling of the vertices [18, 22]). Hence for a bipartite graph G, we let $<'$ be any order that places vertices of A before those of B. The rigidity matrix $R^{(k,k)}(G)$ is then simply the transpose of the matrix obtained from $H^{k,<'}(G)$ by multiplying the B-labeled rows of $H^{k,<'}(G)$ by -1. In particular, $\mathrm{rank}(R^{(k,k)}(G)) = \mathrm{rank}(H^{(k,<')}(G))$. Therefore, we have:

Remark 3.5. A bipartite graph G is (k,k)-stress free if and only if it is k-acyclic in the sense of [19].

Finally, we notice that the notions of (k,l)-stress freeness and (k,l)-rigidity introduced here are equivalent to the geometric notions of Section 1.2. It follows from Proposition 3.3 that G is (k,l)-stress free if and only if the left kernel (i.e., the space of linear dependencies of rows) of the rigidity matrix $R^{(k,l)}(G)$ equals (0). Similarly, by Remark 3.4, G is (k,l)-rigid if and only if $\mathrm{row}(R^{(k,l)}(G)) = \mathrm{row}(R^{(k,l)}(K_{A,B}))$, which happens if and only if $\ker(R^{(k,l)}(G)) = \ker(R^{(k,l)}(K_{A,B}))$. Thus, considering a (k,l)-embedding of G given by $\phi(a) = (\theta_{ia} : 1 \le i \le k) \times (0)$ for $a \in A$ and $\phi(b) = (0) \times (\theta_{i'b} : 1 \le i \le l)$ for $b \in B$, we obtain:

Remark 3.6. A bipartite graph $G = (A \uplus B, E)$ is (k,l)-stress free if and only if for a *generic* (k,l)-embedding ϕ, the condition in eq. (3) holds. A bipartite graph G is (k,l)-rigid if and only if for a generic (k,l)-embedding ϕ, the condition in eq. (4) holds.

We are now in a position to establish the cone, deletion, contraction, and gluing lemmas, paralleling the corresponding statements in classical rigidity.

Lemma 3.7 (Deletion Lemma). *Let G be a bipartite graph, v a vertex of G of degree d, and $G' = G - v$ the graph obtained from G by deleting v.*

1. *If G' is (k,l)-stress free and $d \le \begin{cases} l & \text{if } v \in A \\ k & \text{if } v \in B \end{cases}$, then G is (k,l)-stress free.*
2. *If G' is (k,l)–rigid and $d \ge \begin{cases} l & \text{if } v \in A \\ k & \text{if } v \in B \end{cases}$, then G is (k,l)-rigid.*

Proof. Assume $v \in A$ (the case $v \in B$ is very similar). The matrix $R^{(k,l)}(G)$ is obtained from $R^{(k,l)}(G')$ by adjoining the l columns corresponding to v and the d rows corresponding to the edges containing v. As v is not the end-point of any edge of G', these l new columns consist of zeros followed by a generic $d \times l$ block. Thus,

$$\operatorname{rank}(R^{(k,l)}(G)) \ge \operatorname{rank}(R^{(k,l)}(G')) + \min\{d, l\}. \tag{5}$$

Now, if G' is (k,l)-stress-free and $d \le l$, then by (5) and Proposition 3.3,

$$|E(G)| \ge \operatorname{rank}(R^{(k,l)}(G)) \ge \operatorname{rank}(R^{(k,l)}(G')) + \min\{d, l\} = |E(G')| + d = |E(G)|.$$

Hence $\operatorname{rank}(R^{(k,l)}(G)) = |E(G)|$, and so G is (k,l)-stress free by Proposition 3.3.

Similarly, if G' is (k,l)-rigid and $d \ge l$, then by (5) and Proposition 3.3,

$$\begin{aligned} l|A| + k|B| - kl &\ge \operatorname{rank}(R^{(k,l)}(G)) \ge \operatorname{rank}(R^{(k,l)}(G')) + \min\{d, l\} \\ &= [l(|A|-1) + k|B| - kl] + l = l|A| + k|B| - kl. \end{aligned}$$

Hence $\operatorname{rank}(R^{(k,l)}(G)) = l|A| + k|B| - kl$, and so G is (k,l)-rigid by Proposition 3.3. $\square$

Lemma 3.8 (Contraction Lemma). *Let $G = (V, E)$ be a bipartite graph, v and w two vertices of G that belong to the same part, C the set of common neighbors of v and w, and $G' = (V - \{v\}, E')$ the graph obtained from G by contracting v with w.*

1. *If G' is (k,l)-stress free and $|C| \le \begin{cases} l & \text{if } v \in A \\ k & \text{if } v \in B \end{cases}$, then G is (k,l)-stress free.*
2. *If G' is (k,l)-rigid and $|C| \ge \begin{cases} l & \text{if } v \in A \\ k & \text{if } v \in B \end{cases}$, then G is (k,l)-rigid.*

Proof. For both parts assume that $v, w \in A$ (the case of $v, w \in B$ is analogous), and let M be the matrix obtained from $R^{(k,l)}(G)$ by replacing each θ_{iv} with θ_{iw}, $1 \le i \le k$.

Part 1: By Proposition 3.3, to complete the proof we must show that if $R^{(k,l)}(G')$ has linearly independent rows, then so does $R^{(k,l)}(G)$. As M is a specialization of $R^{(k,l)}(G)$, it suffices to check that the rows of M are linearly independent. And, indeed, if there is a linear dependence among the rows of M, then it induces the same dependence (i.e., with the same coefficients) among the rows of the matrix M' obtained from M by adding the columns of v to the columns of w and deleting the columns of v. However, since G' is obtained from G by contracting v with w, it follows that the matrix M' is obtained from $R^{(k,l)}(G')$ by duplicating $|C|$ rows: for each $c \in C$, the row of $R^{(k,l)}(G')$ labeled by wc appears in M' twice — once labeled by wc and another time by vc. As the rows of $R^{(k,l)}(G')$ are linearly independent, we conclude that each nontrivial dependence among the rows of M is supported on the rows labeled by $\{vc, wc : c \in C\}$. Since $|C| \leq l$, and since the restriction of these $2|C|$ rows to the $2l$ columns of v and w is of the form $\begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$, where Z is a generic $|C| \times l$ matrix, we infer that the rows of M, and hence also of $R^{(k,l)}(G)$ are linearly independent. The assertion of Part 1 follows.

Part 2: According to Proposition 3.3, it suffices to show that if $\mathrm{rank}(R^{(k,l)}(G')) = l(|A|-1) + k|B| - kl$, then $\mathrm{rank}(R^{(k,l)}(G)) = l|A| + k|B| - kl$. Since Remark 3.4 implies that (k,l)-rigidity can be destroyed, but not created by deleting edges, we assume that v, w have exactly l common neighbors, as the extra edges can be deleted. Hence, $|E'| = |E| - l$.

The argument used in Part 1 leads to a stronger statement: if $|C| = l$, then

$$\dim \mathrm{Lker}(R^{(k,l)}(G)) \leq \dim \mathrm{Lker}(M) \leq \dim \mathrm{Lker}(R^{(k,l)}(G')).$$

(Here Lker denotes the left kernel.) Indeed, if $|C| = l$, then the matrix Z from Part 1 is invertible, and so the map sending a vector $(u_e)_{e\in E} \in \mathrm{Lker}(M)$ to $(u'_e)_{e\in E'} \in \mathrm{Lker}(R^{(k,l)}(G'))$, where $u'_{wc} = u_{vc} + u_{wc}$ if $c \in C$ and $u'_e = u_e$ otherwise, is injective. Therefore,

$$\begin{aligned}
\mathrm{rank}(R^{(k,l)}(G)) &= |E| - \dim \mathrm{Lker}(R^{(k,l)}(G)) \\
&\geq |E'| + l - \dim \mathrm{Lker}(R^{(k,l)}(G')) \\
&= \mathrm{rank}(R^{(k,l)}(G')) + l \\
&= l(|A|-1) + k|B| - kl + l \\
&= l|A| + k|B| - kl,
\end{aligned}$$

and the result follows. □

The following lemma is a bipartite analog of the gluing lemma [2, Theorem 2] (in the plane) and [46, Lemma 11.1.9] (for the general case) that treats generic rigidity for the union of general graphs.

Lemma 3.9 (Gluing Lemma). *Let $G = (A \uplus B, E)$ be a bipartite graph written as the union $G = G_1 \cup G_2$ of two bipartite graphs $G_1 = (A_1 \uplus B_1, E_1)$ and $G_2 = (A_2 \uplus B_2, E_2)$.*

1. *If G_1 and G_2 are (k,l)-rigid, $|A_1 \cap A_2| \geq k$, and $|B_1 \cap B_2| \geq l$, then G is (k,l)-rigid.*
2. *If G_1 and G_2 are (k,l)-stress free, and $G_1 \cap G_2$ is (k,l)-rigid, then G is (k,l)-stress free.*
3. *If G_1 and G_2 are (k,l)-stress free, and either $|A_1 \cap A_2| \leq k$ and $|B_1 \cap B_2| = 0$, or $|A_1 \cap A_2| = 0$ and $|B_1 \cap B_2| \leq l$, then G is (k,l)-stress free.*

Proof. To prove Part 1, by Remark 3.4 we may assume that G_1 and G_2 are complete bipartite graphs. Construct G from G_1 by adding the vertices of $G_2 \setminus G_1$ one by one; when adding a vertex v add also the edges in G between v and the former vertices (namely, the vertices of G_1 and the vertices of $G_2 \setminus G_1$ that were added before v). Note that since $|A_1 \cap A_2| \geq k$ and since each vertex $v \in B_2$ is connected to all vertices of

$A_1 \cap A_2$, every time we add a vertex $v \in B_2 \setminus B_1$, we add it as a vertex of degree at least k. Similarly, every time we add a vertex $v \in A_2 \setminus A_1$, we add it as a vertex of degree at least l. Since G_1 is (k,l)-rigid, the Deletion Lemma (Lemma 3.7) combined with induction implies that all graphs in this sequence, including G, are (k,l)-rigid.

To prove Parts 2 and 3, consider the spaces $\mathrm{row}(R^{(k,l)}(G_i))$, $\mathrm{row}(R^{(k,l)}(K_{A_i,B_i}))$ $(i = 1,2)$ as well as $\mathrm{row}(R^{(k,l)}(G_1 \cap G_2))$ and $\mathrm{row}(R^{(k,l)}(K_{A_1,B_1} \cap K_{A_2,B_2}))$ as subspaces of $\mathbb{R}^{l|A|+k|B|}$. Note that under the conditions of either of the Parts 2 and 3

$$\mathrm{row}(R^{(k,l)}(G_1 \cap G_2)) = \mathrm{row}(R^{(k,l)}(K_{A_1,B_1} \cap K_{A_2,B_2})).$$

In the case of Part 2, this follows from the (k,l)-rigidity of $G_1 \cap G_2$, and in the case of Part 3, from the equality of graphs: $K_{A_1,B_1} \cap K_{A_2,B_2} = G_1 \cap G_2$ (indeed, both graphs are edgeless graphs on the same number of vertices).

Our proof relies on Lemma 3.10 below. As $\mathrm{row}(R^{(k,l)}(G_i)) \subseteq \mathrm{row}(R^{(k,l)}(K_{A_i,B_i}))$ (for $i = 1,2$), Lemma 3.10 yields

$$\begin{aligned}
\mathrm{row}(R^{(k,l)}(G_1)) \cap \mathrm{row}(R^{(k,l)}(G_2)) &\subseteq \mathrm{row}(R^{(k,l)}(K_{A_1,B_1})) \cap \mathrm{row}(R^{(k,l)}(K_{A_2,B_2})) \\
&= \mathrm{row}(R^{(k,l)}(K_{A_1\cap A_2,B_1\cap B_2})) \\
&= \mathrm{row}(R^{(k,l)}(G_1 \cap G_2)).
\end{aligned}$$

Assume now that the rows $(R_e : e \in E)$ of $R^{(k,l)}(G)$ satisfy an $\mathbb{R}$-linear dependence

$$\sum_{e\in E_1} \alpha_e R_e = \sum_{e\in E_2\setminus E_1} \alpha_e R_e. \tag{6}$$

Since the left-hand side of (6) is evidently in $\mathrm{row}(R^{(k,l)}(G_1))$ and the right-hand side is in $\mathrm{row}(R^{(k,l)}(G_2))$, the previous inclusion implies that the expression on the left-hand side of (6) is in the row span of $R^{(k,l)}(G_1 \cap G_2)$. Thus, the left-hand side of (6) can be rewritten using only edges $e \in E_1 \cap E_2$, and as G_2 is (k,l)-stress free, all the coefficients on the right-hand side of (6) are zeros. Then, as G_1 is (k,l)-stress free, all the coefficients on the left-hand side of (6) are zeros as well. Hence, G is (k,l)-stress free. ☐

For $i \in [l], a \in A$, let $e_{i,a}$ denote the unit vector of $\mathbb{R}^{l|A|+k|B|}$ with the coordinate 1 in the ith of the l slots allotted for a and zeros everywhere else. Define $e_{j,b'}$ for $j \in [k], b' \in B$ similarly. Using this notation, the row of $R^{(k,l)}(K_{A,B})$ corresponding to the edge ab' can be written as $\sum_{i=1}^{l} \theta_{i'b'} e_{i,a} + \sum_{j=1}^{k} \theta_{ja} e_{j,b'}$.

To finish the proof of the Gluing Lemma, it only remains to verify the following.

Lemma 3.10. *Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ be finite sets such that either*

(i) $|A_1 \cap A_2| \geq k$ *and* $|B_1 \cap B_2| \geq l$, *or*
(ii) $|A_1 \cap A_2| \cdot |B_1 \cap B_2| = 0$, $|A_1 \cap A_2| \leq k$, *and* $|B_1 \cap B_2| \leq l$.

Let $\mathcal{V}_1 = \mathrm{row}(R^{(k,l)}(K_{A_1,B_1}))$, $\mathcal{V}_2 = \mathrm{row}(R^{(k,l)}(K_{A_2,B_2}))$, $\mathcal{V}_\cap = \mathrm{row}(R^{(k,l)}(K_{A_1\cap A_2,B_1\cap B_2}))$ be three vector spaces considered as subspaces of $\mathbb{R}^{l|A|+k|B|}$. Then $\mathcal{V}_1 \cap \mathcal{V}_2 = \mathcal{V}_\cap$.

Proof. We first treat case (i). It suffices to show that $\mathcal{V}_1^\perp + \mathcal{V}_2^\perp = \mathcal{V}_\cap^\perp$, where $\mathcal{V}^\perp$ denotes the orthogonal complement of $\mathcal{V}$ in $\mathbb{R}^{l|A|+k|B|}$ (equivalently, it denotes the kernel of the corresponding matrix). We will do this by explicitly computing $\mathcal{V}_1^\perp$, $\mathcal{V}_2^\perp$, and $\mathcal{V}_\cap^\perp$.

For $r \in [l]$ and $p \in [k]$, define $w^{rp} \in \mathbb{R}^{l|A|+k|B|}$ by $w^{rp} = \sum_{a\in A} \theta_{pa} e_{r,a} - \sum_{b'\in B} \theta_{r'b'} e_{p,b'}$, where r' is the element of $[l']$ corresponding to r in $[l]$. Note that w^{rp} is orthogonal to all rows of $R^{(k,l)}(K_{A,B})$, and hence also to all elements of $\mathcal{V}_1$. Thus

$$\mathcal{B}_1 := \{w^{rp} : r \in [l], p \in [k]\} \cup \{e_{i,a} : i \in [l], a \in A\setminus A_1\} \cup \{e_{j,b'} : j \in [k], b' \in B\setminus B_1\} \subset \mathcal{V}_1^\perp.$$

Moreover, the vectors of $\mathcal{B}_1$ are linearly independent: indeed, using the unit vectors appearing in the above union, we only need to check that the set $\{w_1^{rp} := \sum_{a\in A_1} \theta_{pa} e_{r,a} - \sum_{b'\in B_1} \theta_{r'b'} e_{p,b'} : r \in [l], p \in [k]\}$ is linearly independent. However, restricting the matrix formed by these kl row vectors to the columns of the first k vertices in A_1 already yields an invertible $kl \times kl$ matrix, as Gauss elimination shows.

Since $|A_1 \cap A_2| \geq k$ and $|B_1 \cap B_2| \geq l$, the graph K_{A_1,B_1} is (k,l)-rigid. Hence $\dim \mathcal{V}_1^\perp = kl + l(|A| - |A_1|) + k(|B| - |B_1|) = |\mathcal{B}_1|$, and we obtain that $\mathcal{B}_1$ is a basis of $\mathcal{V}_1^\perp$. The same reasoning leads to analogous expressions for bases $\mathcal{B}_2$ and $\mathcal{B}_\cap$ of $\mathcal{V}_2^\perp$ and $\mathcal{V}_\cap^\perp$, respectively. The result follows since $\mathcal{B}_1 \cup \mathcal{B}_2 = \mathcal{B}_\cap$.

In case (ii), we must show that $\mathcal{V}_1 \cap \mathcal{V}_2 = (0)$. As a warm-up, if $|A_1 \cap A_2| = |B_1 \cap B_2| = 0$, then the above description of $\mathcal{B}_1$ and $\mathcal{B}_2$ yields that $\mathcal{B}_1 \cup \mathcal{B}_2$ is a spanning set for $\mathbb{R}^{l|A|+k|B|}$, and hence completes the proof.

If, say, $|A_1 \cap A_2| \leq k$ and $B_1 \cap B_2 = \emptyset$, then by definition of $R^{(k,l)}(G)$

$$\mathcal{V}_1 \cap \mathcal{V}_2 \subseteq \mathrm{Span}\{e_{i,a} : i \in [l],\ a \in A_1 \cap A_2\}. \tag{7}$$

However, since for a fixed $r \in [l]$, the k scalar products (where p ranges over $[k]$)

$$\langle w^{rp}, \sum_{i\in[l]} \sum_{a\in A_1\cap A_2} \alpha_{ia} e_{i,a} \rangle = \sum_{a\in A_1\cap A_2} \alpha_{ra}\theta_{pa}$$

vanish simultaneously only if $\alpha_{ra} = 0$ for all a, we infer that no nonzero vector from the right-hand side of eq. (7) is orthogonal to all w^{rp}. Thus $\mathcal{V}_1 \cap \mathcal{V}_2 = (0)$, as required. (The case of $|B_1 \cap B_2| \leq l$ and $A_1 \cap A_2 = \emptyset$ is treated similarly.) $\square$

We finish this section with the Cone lemma. This will require the following definition.

Definition 3.11. Let $G = (A \uplus B, E)$ be a bipartite graph, where $A = [n]$ and $B = [m']$. Let $A^* := A \cup \{0\}$ and $B^* := B \cup \{0'\}$. The *left-side cone over* G, $C^L G$, is the bipartite graph with the vertex set $A^* \uplus B$ and the edge set $E \cup \{0b' : b' \in B\}$. The *right-side cone over* G, $C^R G$, is the bipartite graph with the vertex set $A \uplus B^*$ and the edge set $E \cup \{a0' : a \in A\}$.

To compute the balanced shifting of $C^L G$, we extend our order $<$ on V to an order $<_0$ on $A^* \cup B$ by requiring that 0 is the smallest vertex. Similarly, to work with $C^R G$, we extend $<$ to an order $<_{0'}$ on $A \cup B^*$ by requiring that $0'$ is the smallest vertex. Note that if $<$ is (k,l)-admissible, then $<_0$ is $(k+1,l)$-admissible and $<_{0'}$ is $(k,l+1)$-admissible.

Lemma 3.12 (Cone Lemma)**.** *The operations of coning and shifting commute, that is,*

$$(C^L G)^{b,<_0} = C^L(G^{b,<}) \quad \textit{and} \quad (C^R G)^{b,<_{0'}} = C^R(G^{b,<}).$$

Thus, G is (k,l)-rigid if and only if $C^L G$ is $(k+1,l)$-rigid (equivalently, if and only if $C^R G$ is $(k,l+1)$-rigid), and G is (k,l)-stress free if and only if $C^L G$ is $(k+1,l)$-stress free (equivalently, if and only if $C^R G$ is $(k,l+1)$-stress free).

Proof. The proof that coning and shifting commute is very similar to that of the cone lemma for the case of symmetric shifting, see [5, Lemma 3.3(4)]. We omit the details. The second statement then follows from the combinatorial definitions of "rigid" and "stress-free" in Definition 3.1. ☐

4. Bipartite planar graphs

In this section we establish a bipartite analog of the rigidity criterion for planar graphs. Recall that according to Proposition 1.6, for a graph G, the existence of K_5 in G^s is an obstruction to the planarity of G. Here we show that for a bipartite G, the existence of $K_{3,3}$ in $G^{b,<}$ (where $<$ is a $(2,2)$-admissible order) is also an obstruction to the planarity of G, that is, we prove the following more precise version of Theorem 1.8:

Theorem 4.1. *If G is a planar bipartite graph and $<$ is a $(2,2)$-admissible order, then $K_{3,3}$ is not a subgraph of $G^{b,<}$. Equivalently, planar bipartite graphs are $(2,2)$-stress free.*

Our proof of Theorem 4.1 can be considered as a bipartite analog of Whiteley's proof [45] of Gluck's result. It relies on the lemmas established in the previous section as well as on some combinatorial properties of bipartite planar graphs. The first such property is a bipartite analog of the fact that any maximal planar graph with at least 3 vertices partitions the 2-sphere into triangles; the second is a bipartite analog of the the fact that maximal planar graphs on n vertices have $3n-6$ edges. Both properties are well-known and included here only for completeness.

A planar bipartite graph is *maximal* if the addition of any new edge (but no new vertices) results in a graph that is either non-planar or non-bipartite.

Lemma 4.2. *If $G = (A \uplus B, E)$ is a maximal planar bipartite graph, where $|A|, |B| \geq 2$, then G partitions the 2-sphere into 2-cells whose boundaries are 4-gons.*

Proof. Consider a planar drawing of G. If G has a vertex of degree 0 or 1, then G is not maximal. Thus we can assume that all degrees are at least 2, and hence that each edge is incident with two 2-cells of the 2-sphere. If one of the cells is not a 4-gon, it has at least 6 vertices, say, $(a, b, c, d, e, f, \ldots)$ in this order along its boundary. By planarity of the drawing, not both ad and be are edges of G (drawn outside of this 2-cell). Since adding such a missing edge and drawing it inside this 2-cell preserves bipartiteness and planarity, it follows that G is not maximal. ☐

Lemma 4.3. *If $G = (A \uplus B, E)$ is a maximal planar bipartite graph on N vertices, where $|A|, |B| \geq 2$, then G has $2N-4$ edges. Thus, if a planar bipartite G with $N \geq 4$ vertices has $2N-4$ edges, then G is maximal.*

Proof. Let e and c be the number of edges and 2-cells of G, respectively. Each 2-cell has 4 edges and each edge is contained in two 2-cells. Thus $c = e/2$. By the Euler formula $N + c - e = 2$, and so $e = 2N - 4$. □

The following result will allow us to invoke the Contraction Lemma.

Lemma 4.4. *Let G be a maximal planar bipartite graph on N vertices, $N > 4$. Then every 2-cell induced by a planar drawing of G has a pair of opposite vertices with exactly two common neighbors, namely, the other two vertices on the boundary of this 2-cell. Assume v, w form such a pair. Then the graph G' obtained from G by contracting v with w is a maximal planar bipartite graph on $N - 1$ vertices.*

Proof. To prove the first assertion, note that if (v, b, w, c) is a 4-cycle in a planar drawing of G such that v, w have another common neighbor a, and b, c have another common neighbor x, then exactly one of a, x is inside the cycle and the other outside. In particular, (v, b, w, c) does not bound a 2-cell.

Let v, w be a pair guaranteed by the first part. Deleting v from the drawing of G creates one new 2-cell X, with boundary cycle $(w, c, x_1, y_1, x_2, ..., y_{k-1}, x_k, d, w)$, where c, d are the common neighbors of v, w in G. To obtain a planar drawing of G', draw the edges wy_i (replacing the edges vy_i) inside the cell X according to this order. The graph G' has one vertex and two edges fewer than G (indeed, the vertex v and the edges vc and wd are "lost"), and so by Lemma 4.3, G' is maximal. □

We are now in a position to prove Theorem 4.1.

Proof. Let $G = (V, E)$ be as in the theorem. Since G^b is bipartite with the same vertices as G on each side, it follows that if G has a side with at most one vertex, then $K_{3,3} \not\subseteq G^{b,<}$ for any order $<$. Thus assume that G has at least 2 vertices on each side.

It is easy to see from the definition of balanced shifting that if $H = (A \uplus B, E_H)$ is a subgraph of $G = (A \uplus B, E)$, then $H^{b,<}$ is a subgraph of $G^{b,<}$. (This follows from the fact that the Stanley-Reisner ideals of H and G satisfy $I_H \supseteq I_G$.) Hence, we assume without loss of generality that G is maximal. We prove by induction on the size of V that such G is $(2, 2)$-stress free (and also $(2, 2)$-rigid). The base case, namely $|V| = 4$, does hold as in this case $G = G^b = C_4$. Thus assume $|V| > 4$, and consider vertices v, w of G as in Lemma 4.4. Let G' be the graph obtained by contracting v with w. Lemma 4.4, the induction hypothesis, and the Contraction Lemma (Lemma 3.8) complete the proof. □

In view of Proposition 1.6, a remaining natural problem is to find a notion of a minor for bipartite graphs, denoted $<_b$, for which $K_{3,3} <_b G$ would imply that G is not planar, and $K_{3,3} \subseteq G^b$ would imply that $K_{3,3} <_b G$. In a separate paper joint with Chudnovsky and Seymour [8], we propose such a notion of minors, $<_b$, and prove a bipartite analog of Wagner's planarity criterion: *A bipartite graph G is planar if and only if $K_{3,3} \not<_b G$.*

A notion closely related to planarity of graphs is that of *linkless embeddability* of graphs in $\mathbb{R}^3$. A graph G is called *linklessly embeddable* if there is an embedding of G

into $\mathbb{R}^3$ in such a way that every two cycles of G have zero linking number. (A subfamily of linklessly embeddable graphs is that of *apex graphs*: graphs that can be made planar by the removal of a single vertex.) It is a theorem of Sachs [34] that $K_{4,4}$ minus an edge, which we denote by $K_{4,4}^-$, is not linklessly embeddable. This fact and Theorem 4.1 lead us to the following conjecture.

Conjecture 4.5. *Let $G = (A \uplus B, E)$ be a bipartite linklessly embeddable graph and $<$ a $(3,3)$-admissible order. If $|A|, |B| \geq 4$ then $K_{4,4}^-$ is not a subgraph of G^b, and thus $E \leq 3|V(G)| - 10$. In particular, all bipartite linklessly embeddable graphs are $(3,3)$-stress free; hence if A and B are each of size at least 3, then $E \leq 3|V(G)| - 9$.*

At present, even the inequality $E \leq 3|V(G)| - 9$ of the above conjecture is open. (Equality $|E| = 3|V| - 9$ holds for complete bipartite graphs $K_{3,m}$; these graphs are apex graphs, and hence linklessly embeddable.) As for the rest of the conjecture, the following weaker statement is easy to prove.

Lemma 4.6. *Let G be a linklessly embeddable bipartite graph. Then G is $(7,7)$-stress free, and so $G^{b,<}$ does not contain $K_{8,8}$, where $<$ is any $(7,7)$-admissible order.*

Proof. Note that G does not contain K_6 as a minor; this is an easy part of the forbidden minor characterization of linklessly embeddable graphs [33]. Thus, by a result of Mader [26], if G has N vertices then G has fewer than $4N$ edges. Hence there is a vertex v in G whose degree is at most 7. Now we use the Deletion Lemma (Lemma 3.7) and induction to conclude that $G - v$ is $(7,7)$-stress free, and hence so is G. $\square$

Some other remarkable phenomena from graph rigidity theory can be considered in the context of bipartite rigidity. First, recall that Gluck's proof of the generic rigidity of maximal planar graphs is based on the theorems of Steinitz and Dehn–Alexandrov. Steinitz's theorem asserts that every maximal planar graph is the graph of some 3-dimensional simplicial polytope, while the Dehn–Alexandrov theorem posits that the graph of any 3-dimensional simplicial polytope is infinitesimally rigid. We do not know if the results of Dehn and Alexandrov have bipartite analogs. Second, the non-generic embeddings of maximal planar triangulations for which infinitesimal rigidity fails are also quite fascinating (e.g., in view of Bricard's Octahedra and Connelly's flexible spheres [9]). The analogous situation for infinitesimal $(2,2)$-rigidity of bipartite planar quadrangulations is also very interesting.

5. Laman-type results for bipartite graphs

In this section we apply the theory developed so far to bipartite trees and outerplanar graphs. We also consider a bipartite analog of the Laman theorem. This theorem, see [24], provides a combinatorial characterization of minimal (with respect to deletion of edges) generically 2-rigid graphs:

Theorem 5.1 (Laman)**.** *A graph $G = (V, E)$ is minimal generically 2-rigid if and only if the following conditions hold:*

(i) $|E| = 2|V| - 3$, *and*
(ii) *every induced subgraph* $G[V'] = (V', E')$ *with* $|V'| \geq 2$ *satisfies* $|E'| \leq 2|V'| - 3$.

Inspired by this result, we consider the relation between (k,l)-rigidity and the following combinatorial condition, analogous to the above Laman condition.

Definition 5.2. A bipartite graph $G = (A \biguplus B, E)$ with $|A| \geq k$ and $|B| \geq l$ is called (k,l)-*Laman* if

(i) $|E| = l|A| + k|B| - kl$, and
(ii) every induced subgraph $G[V'] = (A' \biguplus B', E')$ of G with $|A'| \geq k$ and $|B'| \geq l$ has at most $l|A'| + k|B'| - kl$ edges.

We say that a (k,l)-rigid graph G is (k,l)-*minimal* if the deletion of an arbitrary edge of G results in a graph that is not (k,l)-rigid. By Proposition 3.3, G is (k,l)-minimal if and only if G is (k,l)-rigid and stress free. The following implication holds:

Proposition 5.3. *If a graph G is (k,l)-minimal then G is (k,l)-Laman.*

Proof. As G is (k,l)-minimal, $\text{rank}(R^{(k,l)}(G)) = l|A| + k|B| - kl = |E|$, where the second equality holds by the minimality of G. Hence condition (i) of Definition 5.2 is satisfied. If condition (ii) is violated for some induced subgraph $G[V']$, then the rows of $R^{(k,l)}(G[V'])$ are linearly dependent, also when viewed as rows of $R^{(k,l)}(G)$, contradicting the fact that G is (k,l)-minimal, and hence, in particular, (k,l)-stress free. $\square$

What about the converse statement? It follows from Whiteley's paper [44] that the converse does hold if one of k,l is equal to 1:

Theorem 5.4. *For any $k \geq 1$, if a graph G is $(k,1)$-Laman then G is $(k,1)$-minimal.*

Proof. In [44, Def. 4.2], Whiteley considers the following specialization of $R^{(k,1)}(G)$: for all $b \in B$, $\theta_{1'b}$ is specialized to 1, and for all $a \in A$, θ_{ka} is specialized to 1. He then proves [44, Thm. 4.1] that if G is $(k,1)$-Laman then the rank of the resulting matrix is $|E|$, and hence so is the rank of $R^{(k,1)}(G)$. $\square$

On the other hand, as the following example shows, the converse of Proposition 5.3 is false when both $k,l \geq 2$.

Example 5.5.

1. Let G_1 and G_2 be two copies of $K_{3,3}$ minus an edge, and let G be obtained by gluing G_1 and G_2 along the two vertices of the missing edge, denoted ab. Then $G = (A \biguplus B, E)$ is $(2,2)$-Laman, but G is not $(2,2)$-stress free, and hence is not $(2,2)$-minimal.
2. For $k,l \geq 2$, let
$$H = \underbrace{C^R \ldots C^R}_{l-2} \underbrace{C^L \ldots C^L}_{k-2} G$$
be obtained from G by iterative coning. Then H is (k,l)-Laman, but it is not (k,l)-minimal.

Proof. As G_1^b and G_2^b are both isomorphic to $K_{3,3}$ minus the edge between 3 and $3'$, it follows that each of G_1, G_2 is $(2,2)$-rigid. Hence in the rigidity matrix $R^{(2,2)}(G)$, there is a non-trivial linear combination of the rows of G_1 that equals the row of the missing edge ab, and similarly for G_2; the difference of these two linear combinations provides a non-zero linear dependence of the rows of $R^{(k,l)}(G)$. Thus G is not $(2,2)$-stress free, and hence it is is not $(2,2)$-minimal. On the other hand, one readily checks that G is $(2,2)$-Laman. This completes the proof of Part 1.

Part 2 is an immediate consequence of Part 1. Indeed, the Cone Lemma (Lemma 3.12) and the fact that G is not $(2,2)$-minimal yield that H is not (k,l)-minimal. Further, it is straightforward to check that the left cone over an (r,s)-Laman graph is $(r+1,s)$-Laman while the right cone over an (r,s)-Laman graph is $(r,s+1)$-Laman. As G is $(2,2)$-Laman, we then conclude that H is (k,l)-Laman. □

It would be interesting to have a complete combinatorial characterization of minimal (k,l)-rigid graphs (i.e., bases of the (k,l)-rigidity matroid) even for $k = l = 2$.

We now consider the effect of balanced shifting on trees (which are bipartite) and bipartite outerplanar graphs.

Theorem 5.6. *Let G be a bipartite graph with sides A and B, and let $<$ be the total order on $A \cup B$ with respect to which G^b is computed.*

1. *If $<$ is $(1,1)$-admissible and G is a forest then $K_{2,2} \not\subseteq G^b$; equivalently, G is $(1,1)$-stress free.*
2. *If $<$ is $(2,1)$-admissible and G is outerplanar, then $K_{3,2} \not\subseteq G^b$; equivalently, G is $(2,1)$-stress free. (Similarly, if $<$ is $(1,2)$-admissible and G is outerplanar, then G is $(1,2)$-stress free.)*
3. *If $<$ is $(2,2)$-admissible and G is planar, then $K_{3,3} \not\subseteq G^b$; equivalently, G is $(2,2)$-stress free.*

Proof. Part (1) is proved by induction: for the inductive step, pick a leaf and use the Deletion Lemma (Lemma 3.7). Part (3) is Theorem 4.1. In Part (2), the order $<$ has exactly two vertices in A among the least three vertices. Add to B a new and smallest vertex, and connect it to all vertices of A. This creates a bipartite planar graph and a $(2,2)$-admissible order $<_{0'}$. Now the Cone Lemma (Lemma 3.12) and Part (3) complete the proof. □

We remark that Parts (1) and (2) of Theorem 5.6 also follow easily by counting the edges of induced subgraphs and using Whiteley's criterion, Theorem 5.4.

6. Graphs of polytopes

6.1. **Cubical polytopes.** We now discuss potential applications of bipartite rigidity, á la Kalai [20], to lower bound conjectures on face numbers of cell complexes with a bipartite 1-skeleton.

Recall that by a result of Blind and Blind [6], if P is a cubical d-polytope with $d > 2$, then the graph $G(P)$ of P is bipartite. Moreover, if $d > 2$ is even, then the two sides of $G(P)$ have the same number of vertices. (These results were generalized to arbitrary cubical spheres by Babson and Chan [3].) We are interested in the cubical conjecture of Jockusch [16], see Conjecture 1.10, asserting that if K is a cubical polytope of dimension $d \geq 3$ with $f_0(K)$ vertices and $f_1(K)$ edges, then

Note that if G is $(2, d-1)$-rigid, and has the same number of vertices on each side, then G has at least $\frac{d+1}{2} f_0(G) - 2(d-1)$ edges. The graph $G(P)$ of a stacked cubical polytope P is bipartite and has the same number of vertices on each side, but has only $\frac{d+1}{2} f_0(P) - 2^{d-1}$ edges. We will show in Proposition 6.5 that for such P, it is possible to add to $G(P)$ exactly $2^{d-1} - 2(d-1)$ edges, all in one facet of P, in such a way that the resulting graph is $(2, d-1)$-rigid and stress free. We will also establish a similar statement with respect to $(1, d)$-rigidity and stress freeness, see Proposition 6.7.

This yields the following approach to Jockusch's conjecture; specifically, a positive answer to the following problem will imply Conjecture 1.10 for all *even* $d > 2$:

Problem 6.1. *Let G be the graph of a cubical d-polytope, where $d > 2$ is even. Is it possible to add $2^{d-1} - 2(d-1)$ edges to G to obtain a $(2, d-1)$-rigid graph? Is it possible to add $2^{d-1} - d$ edges to G to obtain a $(1, d)$-rigid graph?*

A similar reasoning shows that a positive answer to the following problem will imply Conjecture 1.10 for an arbitrary d:

Problem 6.2. *Let G be the graph of a cubical d-polytope, where $d > 2$. Is it possible to add $2^{d-1} - \lfloor \frac{d+1}{2} \rfloor \lceil \frac{d+1}{2} \rceil$ edges to G to obtain a $(\lfloor \frac{d+1}{2} \rfloor, \lceil \frac{d+1}{2}) \rceil$-rigid graph?*

We are now in a position to show how to add edges to the graph of a *stacked cubical polytope* to make it $(2, d-1)$-rigid and stress-free. (Recall that a stacked cubical polytope is a polytope obtained starting with a cube and repeatedly gluing cubes onto facets.) Our construction relies on the following lemmas.

Lemma 6.3. *For $d \geq 3$, let C^d be the d-cube, and let A and B be the two sides of the bipartite graph $G(C^d)$ of C^d. Fix vertices $v, v^* \in A$ that are contained in a 2-face of C^d. Let F and F^* be opposite facets of C^d such that $v \in F \cap A$ and $v^* \in F^* \cap A$ (they exist when $d \geq 3$). Add to $G(C^d)$ all the edges vb and v^*b^* where $b \in F \cap B$ and $b^* \in F^* \cap B$ to obtain a new graph $G'(C^d)$. Then $G'(C^d)$ is $(2, d-1)$-rigid and stress free.*

Proof. The proof is by induction on d. In the case of $d = 3$ no edges are added and $(2, 2)$-stress freeness follows from Theorem 4.1 (or check directly). Moreover, since the graph of the 3-cube is a *maximal* planar bipartite graph, it is also $(2, 2)$-rigid.

Assume $d > 3$. Then there exists a facet H of C^d containing both v and v^*; we let H^* denote the opposite facet. We now show, in four steps, that $(2, d-1)$-rigidity and stress freeness of $G'(C^d)$ follow from $(2, d-2)$-rigidity and stress freeness of $G'(H) \cong G'(C^{d-1})$ — the graph formed from the graph of H in the same manner as $G'(C^d)$ is formed from the graph of C^d.

Step 1: linearly order all $t \in A \cap F \cap H^*$ and contract successively t with v (in $G'(C^d)$); similarly, for $t^* \in A \cap F^* \cap H^*$ contract successively t^* with v^*; call the resulting graph

G_1. By construction of $G'(C^d)$, v is connected to *every* vertex in $F \cap B$ but only to one vertex in $F^* \cap B$ (namely, the common neighbor of v, v^* in $F^* \cap B$) — the vertex that has no neighbors in $F \cap A$ except v. On the other hand, it is evident from the structure of $G(C^d)$ that t has exactly $d-1$ neighbors in $F \cap B$. Therefore, it follows that t, v have exactly $d-1$ common neighbors in $G'(C^d)$. (The same argument also applies to t^*, v^*.) Hence, by the Contraction Lemma (Lemma 3.8), to complete the proof, it is enough to show that G_1 is $(2, d-1)$-rigid and stress free.

Step 2: successively contract pairs of vertices in $B \cap F \cap H^*$ until a single vertex p remains, and similarly contract vertices in $B \cap F^* \cap H^*$ until a single vertex p^* remains; call the resulting graph G_2. At each contraction, the two identified vertices have exactly two common neighbors, namely, v and v^*. Thus, by Lemma 3.8, it suffices to prove that G_2 is $(2, d-1)$-rigid and stress free.

Step 3: contract p^* with p to obtain G_3. As p and p^* have two common neighbors in G_2, namely v and v^*, by Lemma 3.8 it remains to verify the assertion for G_3.

Step 4: in G_3, p is a right-cone vertex: indeed, it is connected to all vertices that belong to side A. Delete p and all edges incident with it to obtain G_4. By the Cone Lemma (Lemma 3.12) we need to show that G_4 is $(2, d-2)$-rigid and stress free.

It remains to notice that G_4 is obtained from the graph $G(H)$ of H by adding all edges vb where b is a vertex in the facet $H \cap F$ of H, and all edges v^*b^* where b^* is a vertex in the opposite facet $H \cap F^*$ of H. Thus, the assertion follows by induction. □

Lemma 6.4. *For $d \geq 3$, let C^d be the d-cube, let A and B be the two sides of the bipartite graph $G(C^d)$ of C^d, and let F be a facet of C^d. Fix two vertices $u, w \in F \cap A$, and add to $G(C^d)$ all the edges ub and wb where $b \in F \cap B$ to obtain a new graph G. Then G is $(2, d-1)$-rigid and stress free.*

Proof. If $d = 3$, then no edges are added and $(2, 2)$-rigidity and stress freeness follow from Theorem 4.1. Thus, assume $d > 3$.

As before, let F^* be the facet of C^d opposite to F, and let H and H^* be two opposite facets of C^d such that $u \in H$ and $w \in H^*$. For a vertex $b \in B \cap F^*$ denote by b_F the unique neighbor of b in $A \cap F$.

Step 1: for every $b \in B \cap F^* \cap H$ contract b_F with u, and for every $b \in B \cap F^* \cap H^*$ contract b_F with w; call the resulting graph G_1. At each contraction, the two identified vertices have $d-1$ common neighbors. Thus, by the Contraction Lemma (Lemma 3.8) it is enough to show that G_1 is $(2, d-1)$-rigid and stress free.

Step 2: fix $p \in B \cap F$ and successively contract all other vertices in $B \cap F$ with p to obtain G_2. At each contraction, the two identified vertices have u and w as their only common neighbors. Hence by Lemma 3.8 it suffices to check that G_2 is $(2, d-1)$-rigid and stress free.

Step 3: observe that p is a right-cone vertex in G_2; delete it to obtain G_3. By the Cone Lemma (Lemma 3.12), the result will follow if we show that G_3 is $(2, d-2)$-rigid and stress free.

Step 4: fix two vertices $v \in A \cap F^* \cap H$ and $v^* \in A \cap F^* \cap H^*$ that are contained in a 2-face: such v, v^* exist as $d > 3$. Contract u with v and w with v^* to obtain G_4. Since there are $d-2$ common neighbors at each contraction, by Lemma 3.8, it is enough to show that G_4 is $(2, d-2)$-rigid and stress free. This, however, is an immediate consequence of Lemma 6.3, as, using the notation of that lemma, $G_4 = G'(F^*)$. □

Proposition 6.5. *For $d \geq 3$, let P be a stacked cubical d-polytope, let A, B be the two sides of the bipartite graph of P, and let F be a facet of P. Fix vertices $u, w \in F \cap A$ and add to the graph of P all edges ub and wb where $b \in F \cap B$ to obtain a new graph G. Then G is $(2, d-1)$-rigid and stress free.*

Proof. First, observe that P can be formed by successively stacking cubes in a certain order $C_1, C_2, ..., C_m$ satisfying the condition that F is a facet of C_1: indeed, the graph whose vertices are the cubes C_i and whose edges are between the cubes that are glued along a facet, is a tree, and so any cube can be taken to be the first in the stacking process. Let P_i be the stacked cubical sphere obtained by stacking $C_1, ..., C_i$, and let G_i be the corresponding graph (with the added edges in F). In particular, $G = G_m$. We show by induction that G_i is $(2, d-1)$-rigid and stress free.

For G_1, this follows from Lemma 6.4, and so assume $i > 1$. By induction, G_{i-1} is $(2, d-1)$-rigid, and hence its rigidity matrix has the same rank as the $(2, d-1)$-rigidity matrix of the complete bipartite graph on the same vertex set (with same sides as in G_{i-1}), denoted $K(i-1)$. Thus the $(2, d-1)$-rigidity matrices of G_i and $G_i \cup K(i-1)$ have the same rank. Let G'_i be the restriction of $G_i \cup K(i-1)$ to the vertices of C_i. Then G'_i is the graph of the d-cube with all bipartite edges in one of its facets added. By Lemma 6.4, G'_i is $(2, d-1)$-rigid. Therefore, by the Gluing Lemma (Lemma 3.9), the union $G_i \cup K(i-1) = K(i-1) \cup G'_i$ is $(2, d-1)$-rigid, and hence so is G_i.

Counting the number of edges in the graph G_i with sides $A_i \subseteq A$ and $B_i \subseteq B$ yields

$$|E(G_i)| = (d+1) \cdot i \cdot 2^{d-1} - 2(d-1) = (d-1)|A_i| + 2|B_i| - 2(d-1).$$

Thus, G_i is also $(2, d-1)$-stress free. □

Corollary 6.6. *For $d \geq 3$, the graph of a stacked cubical d-polytope is $(2, d-1)$-stress free.*

Using the numerical condition of Theorem 5.4 on $(1, d)$-minimality, we also establish the following $(1, d)$-analog of Proposition 6.5.

Proposition 6.7. *For $d \geq 4$, let P be a stacked cubical d-polytope and F a facet of P. Then it is possible to add to the graph of P exactly $2^{d-1} - d$ edges, all of them in F, so that the resulting graph is $(1, d)$-Laman, and hence $(1, d)$-rigid and stress free.*

The same proof as in Proposition 6.5 shows that the following two lemmas imply Proposition 6.7.

Lemma 6.8. *For $d \geq 4$, it is possible to add $2^{d-1} - d$ edges to the graph of a $(d-1)$-cube so that the resulting graph is $(1, d)$-Laman.*

Proof. The proof is by induction on d. For $d = 4$, we need to add $2^3 - 4 = 4$ edges to a 3-cube. Adding all long diagonals (there are exactly 4 of them) results in $K_{4,4}$, which is easily checked to be $(1, 4)$-Laman.

For the inductive step, consider two opposite facets F' and F'' of a $(d-1)$-cube P, and assume that we can add $2^{d-2} - (d-1)$ edges to the graph $G(F')$ of F' so that the resulting graph is $(1, d-1)$-Laman, and the same amount of edges to the graph $G(F'')$ of F'' so that the resulting graph is $(1, d-1)$-Laman. Also add $d-2$ arbitrary "bipartite" edges that go between F' and F''. Thus, the total number of added edges is $2(2^{d-2} - (d-1)) + (d-2) = 2^{d-1} - d$. We claim that the graph of P with all the added edges is $(1, d)$-Laman. And indeed, for any subgraph $G = ((A' \cup A'') \uplus (B' \cup B''), E)$ of this graph, where $A' \cup B'$ is a subset of the vertex set of F' and $A'' \cup B''$ of F'', and where we denote by $E(C, D)$ the set of edges connecting C to D, we have

$$\begin{aligned}
|E| &= |E(A', B')| + |E(A'', B'')| + |E(A', B'')| + |E(A'', B')| \\
&\leq [(d-1)|A'| + |B'| - (d-1)] + [(d-1)|A''| + |B''| - (d-1)] \\
&\qquad + [|A'| + |A''|] + (d-2) \\
&= d|A' \cup A''| + |B' \cup B''| - d.
\end{aligned}$$

Some explanation is in order: in the second step, the first two summands follow from the inductive hypothesis, the third summand, $|A'| + |A''|$, is implied by the fact that in the original graph of P, for each vertex of A' there is at most one edge from this vertex to B'', and, similarly, for each vertex of A'' there is at most one edge to B'; finally, the $d-2$ added edges between the two facets contribute the last summand. Further, in the above inequality, equality holds when considering the entire graph, and so this graph is $(1, d)$-Laman. □

Lemma 6.9. *For $d \geq 4$, it is possible to add $2^{d-1} - d$ edges to the graph of a d-cube, all of them in one facet, so that the resulting graph is $(1, d)$-Laman.*

Proof. Let P be a d-cube, and let F' and F'' be a pair of opposite facets of P. Using Lemma 6.8, add $2^{d-1} - d$ edges to the graph of F' to make it $(1, d)$-Laman. We claim that the graph of P together with these added edges is $(1, d)$-Laman. And indeed, for any subgraph $G = ((A' \cup A'') \uplus (B' \cup B''), E)$ of the resulting graph,

$$|E| \leq |E(A', B')| + d|A''| + |B''| \leq d|A' \cup A''| + |B' \cup B''| - d,$$

where in the first inequality the summand $d|A''|$ is justified by the fact that each vertex of A'' has degree d, and hence cannot contribute more than d edges, while the summand $|B''|$ is explained by the fact that there is at most one edge from each vertex of B'' to A'; the second inequality is by Lemma 6.8. Further, in the above inequality, equality holds when considering the entire graph, and hence this graph is $(1, d)$-Laman. □

6.2. **Dual to balanced polytopes.** For relevant terminology on simplicial complexes used below, see Section 7.

Recall that the facet-ridge graph of a pure simplicial complex K is the graph whose vertices are facets of K, and two facets are connected by an edge if they share a common ridge. Recall also that a *combinatorial manifold* (without boundary) of dimension $d-1$ is a simplicial complex whose geometric realization is a $(d-1)$-manifold with an additional restriction that all vertex links are piecewise linear homeomorphic to the boundary of a $(d-1)$-simplex.

Let K be a $(d-1)$-dimensional simplicial complex; assume further that K is a combinatorial manifold with a trivial fundamental group. According to Joswig [17], the facet-ridge graph of K is bipartite if and only if K is balanced. (For $d = 3$ this is a classic result by Ore; for $d = 4$ this result goes back to Goodman and Onishi [14], and to the unpublished work of Deligne, Edwards, MacPherson, and Morgan.) In particular, if P is a balanced simplicial polytope and P^* is a polytope dual to P, then the graph of P^* is bipartite. This graph is also d-regular, and hence has $f_{d-1}(K)$ vertices and $df_{d-1}(K)/2$ edges.

Problem 6.10. *Fix $d \geq 3$. Let K be a $(d-1)$-dimensional balanced simplicial complex and assume that K is a combinatorial manifold (without boundary) with a trivial fundamental group. Let G be the facet-ridge graph of K.*

1. *Is this graph $(\lfloor (d+1)/2 \rfloor, \lceil (d+1)/2 \rceil)$-stress free?*
2. *Assume further that K has no missing facets. Is G a $(k, d-k)$-rigid graph for $1 \leq k \leq d-1$?*

We start with Part (1). As the only 2-dimensional manifold with a trivial fundamental group is a sphere, and as the facet-ridge graph of a 2-dimensional simplicial sphere is planar, it follows from Theorem 4.1 that the answer to Problem 6.10(1) is positive in the case of $d = 3$. Also it is well-known and easy to prove by induction on dimension (by considering vertex links) that the number of facets of a balanced $(d-1)$-dimensional manifold (without boundary) is at least 2^d, for all d. Since $2^{d-1} \geq \lfloor (d+1)/2 \rfloor \lceil (d+1)/2 \rceil$, at least the inequality on the number of edges of G implied by Problem 6.10(1) does hold for all values of d.

Next we discuss Part (2). First, to see that the condition in Part (2) is necessary, let $d \geq 4$ and consider $2d$ copies of the boundary complex of the d-dimensional cross polytope C^* (with a natural d-coloring). We label these copies by $\partial C_0^*, \partial C_1^*, \ldots, \partial C_{2d-1}^*$. As the facet-ridge graph of ∂C_i^* is bipartite, we can refer to facets of ∂C_i^* as belonging to either side A or side B of this graph. Pick $2d-1$ facets $H_1', \ldots, H_{2d-1}'$ of ∂C_0^* that belong to side B (this is possible as there are 2^{d-1} such facets in total), and for each $i = 1, \ldots, 2d-1$, pick a facet H_i of ∂C_i^* that belongs to side A. Now, for each $i = 1, \ldots, 2d-1$ glue the complex ∂C_i^* onto ∂C_0^* by identifying each vertex of H_i with the same color vertex of H_i', and removing the resulting common facet. Denote the complex obtained in this way by K. Thus K is balanced. In fact, K is isomorphic to the boundary complex of a balanced simplicial d-polytope.

Let G be the facet-ridge graph of K. Observe that for each $0 \leq i \leq 2d-1$, ∂C_i^* has 2^d facets for the total number of $2d \cdot 2^d$ facets. Since each gluing described above reduces the total number of facets by 2, the complex K has $2d \cdot 2^d - 2(2d-1)$ facets. Hence G has $d \cdot 2^d - (2d-1)$ vertices on each side. By d-regularity of G, we conclude that G has $d^2 \cdot 2^d - 2d^2 + d$ edges. Note also that according to Proposition 3.3, a $(1, d-1)$-minimal graph on the same vertex set as G has $d(d \cdot 2^d - (2d-1)) - (d-1) = d^2 \cdot 2^d - 2d^2 + 1$ edges.

We claim that G is not $(1, d-1)$-rigid. Indeed, if G were $(1, d-1)$-rigid there would be a way to delete $d-1$ edges of G (corresponding to the ridges of K) so that the resulting graph G' is $(1, d-1)$-minimal, and hence by Proposition 5.3, $(1, d-1)$-Laman. However, since each ridge belongs to only two facets, these deletions affect at most $2(d-1)$ of our cross polytopes; in other words, for some $1 \leq i_0 \leq 2d-1$, no ridges of $\partial C^*_{i_0} - \{H_{i_0}\}$ are deleted. The subgraph of G' induced by the facets of $\partial C^*_{i_0} - \{H_{i_0}\} = A' \cup B'$ violates the $(1, d-1)$-Laman condition: since $|A'| = 2^{d-1} - 1$ and $|B'| = 2^{d-1}$, the number of edges in this subgraph is $d2^{d-1} - d$ while $(d-1)|A'| + |B'| - (d-1) = d2^{d-1} - 2(d-1)$. A similar construction works for every $k = 1, \ldots, d-1$, as well as for $d = 3$ (where more copies of C^* are glued together).

Note that if a bipartite graph $G = (A \cup B, E)$ is $(k, d-k)$-rigid then it has at least $(d-k)|A| + k|B| - k(d-k)$ edges — a quantity that is smaller than the number of edges a d-regular bipartite graph G has. Thus at least the inequality on the number of edges of G implied by Problem 6.10(2) does hold for all $1 \leq k < d$.

We now give a positive answer to Problem 6.10(2) for the case of $d = 3$.

Proposition 6.11. *Let K be a a balanced 2-dimensional simplicial sphere without missing triangles, and let G be the facet-ridge graph of K. Then G is $(1, 2)$-rigid.*

Proof. By Theorem 5.4, it suffices to show that G has a subgraph $G'' = G - \{ab, a'b'\}$ with the following property: for every proper subset $V' = A' \uplus B' \subsetneq V(G)$, with both $A' \subseteq A$ and $B' \subseteq B$ nonempty, the induced subgraph $G''[V']$ has at most $2|A'| + |B'| - 2$ edges. Let $e' = |E(G[V'])|$ and $e'' = |E(G''[V'])|$. Thus, $e'' \leq e'$, and we need to prove that $e'' \leq 2|A'| + |B'| - 2$. There are the following four cases to consider. (The deletion of the two edges from G is used only in the last case, and is described there.)

1. If $|B'| \geq |A'| + 2$, then by 3-regularity of G, $e' \leq 3|A'| \leq 2|A'| + |B'| - 2$, and the result follows.

2. If $|B'| \leq |A'| - 1$, then by 3-regularity of G, $e' \leq 3|B'| \leq 2|A'| + |B'| - 2$, and the result follows.

3. If $|A'| = |B'|$, then by 3-regularity and connectivity of G, $e' \leq 3|A'| - 1 = 2|A'| + |B'| - 1$, and so the only remaining subcase here is the case of $G[V']$ being a 3-regular graph minus an edge. Then $G[V - V']$ is also a 3-regular graph minus an edge. Hence G is the union of these two disjoint induced subgraphs plus *two* additional edges. This however contradicts the fact that G is a 3-vertex connected graph (indeed, G is a graph of a simple 3-dimensional polytope), and hence also a 3-edge connected graph.

4. If $|B'| = |A'| + 1$, then by 3-regularity of G, $e' \leq 3|A'| = 2|A'| + |B'| - 1$, and hence the only remaining subcase here is the case of $e' = 3|A'|$. In this case all neighbors of

A' are in B', hence all neighbors of $B - B'$ are in $A - A'$, which means that G is the union of $G[V']$ and $G[V - V']$ plus three additional edges e_1, e_2, e_3 that connect B' with $A - A'$.

On the level of our complex K, this means that K is the union of two 2-dimensional subcomplexes K' and K'', corresponding to the graphs $G[V']$ and $G[V-V']$, respectively, and their common boundary, $\partial(K')$, consists of e_1, e_2, and e_3. However, as $\partial(K')$ is a union of cycles, it follows that the edges e_1, e_2, e_3 form a cycle, and this cycle must not be a missing triangle in K by assumption. As both A' and B' are nonempty, we infer that K'' is a single triangle, and as $|B'| = |A'| + 1$, this triangle belongs to side A. We can choose the edges ab and $a'b'$ to be disjoint. Then not both of them belong to $\partial(K')$, and so at least one of them is in $G[V']$. Hence, $e'' \leq e' - 1 \leq 2|A'| + |B'| - 2$, and the result follows. □

7. Preliminaries on simplicial complexes

First, we recall some basic definitions related to simplicial complexes, to be used in Section 8. For further background see, for instance, [28]. Next, we motivate the questions considered in Section 8.

A *simplicial complex* K on the *vertex set* V is a collections of subsets of V such that (i) $\{v\} \in K$ for all $v \in V$, and (ii) if $G \subset F$ and $F \in K$, then $G \in K$. The elements of K are called *faces*. In particular, the empty set is a face of K. A set $F \subseteq V$ that is not a face of K but all of whose proper subsets are faces of K is called a *missing face* of K.

For a simplicial complex K and a face σ of K, the *antistar* of σ in K is the subcomplex of K given by $\mathrm{ast}_K(\sigma) = \{\tau \in K : \ \sigma \nsubseteq \tau\}$, and the *link* of σ is the subcomplex $\mathrm{lk}_K(\sigma) = \{\tau \in K : \ \sigma \cap \tau = \emptyset, \ \sigma \cup \tau \in K\}$. The *join* of two simplicial complexes K and L on disjoint vertex sets is $K * L = \{\sigma \cup \tau : \ \sigma \in K, \ \tau \in L\}$. For instance, $[3] * [3]$ is simply $K_{3,3}$, where $[3]$ denotes the complex consisting of three isolated vertices.

The *dimension* of a face σ is defined by $\dim(\sigma) := |\sigma| - 1$; the dimension of a simplicial complex K is defined by $\dim(K) := \max\{\dim(\sigma) \ : \ \sigma \in K\}$. If all maximal (w.r.t. containment) faces of K have the same dimension, then K is *pure*; the top-dimensional faces of K are called *facets* and faces of codimension-1 are called *ridges*. For instance, the collection of all subsets of $[n]$ of size at most i forms a pure $(i-1)$-dimensional simplicial complex that we denote by $\binom{[n]}{\leq i}$.

If the vertices of K can be colored by $\dim(K)+1$ colors in such a way that the vertices of any edge of K receive different colors, then K is *balanced*. For example, bipartite graphs are balanced 1-dimensional complexes. When discussing a balanced complex K, we assume that its vertex set V is endowed with such a coloring: $V = V_1 \uplus \cdots \uplus V_d$, where $d = \dim(K) + 1$. In this situation, for $T = \{i_1, \ldots, i_t\} \subseteq [d]$, we denote by K_T the restriction of K to the vertex set $V_{i_1} \uplus \cdots \uplus V_{i_t}$.

As in the case of graphs, for a simplicial complex K on V one can define the *Stanley-Reisner ring* of K. To do so, consider a variable x_v for every vertex $v \in V$. Let

$S = \mathbb{R}[x_v \ : v \in V]$, let I_K be the ideal of S generated by squarefree monomials corresponding to non-faces of K, and let $\mathbb{R}[K] := S/I_K$.

Also, as in the case of bipartite graphs, for a *balanced* $(d-1)$-dimensional simplicial complex K on $V = V_1 \uplus \cdots \uplus V_d$, where V_i denotes the set of vertices of color i, we can use the Stanley-Reisner ring of K to define a balanced shifting of K, K^b. To do so, one needs a total order $<$ on V that extends given total orders on each of $V_1, \ldots, V_d$, as well as a block matrix $\Theta = \Theta_1 \times \cdots \times \Theta_d \in \mathrm{GL}_{|V_1|}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{|V_d|}(\mathbb{R})$, where $\Theta_1, \ldots, \Theta_d$ are generic matrices. The rest of the definition is analogous to that for graphs: for $v \in V_i$, set $\deg x_v := e_i \in \mathbb{Z}^d$ (here e_i is the ith unit vector); this makes $\mathbb{R}[K]$ into a $\mathbb{Z}^d$-graded ring. Now, for each $T \subseteq [d]$, let $e_T = \sum_{i \in T} e_i \in \mathbb{Z}^d$, and define $\mathcal{B}_T$ to be the greedy lexicographic basis (w.r.t. $<$) of the vector space $\mathbb{R}[K]_{e_T}$ chosen from the monomials written in θ's. Let $\mathcal{B} = \cup_{T \subseteq [d]} \mathcal{B}_T$. Finally, define K^b as a collection of subsets of V that are supports of monomials from $\mathcal{B}$. It is shown in [4] that K^b is a balanced simplicial complex; it has the same f-vector as K; moreover, K^b is balanced-shifted: if $v \in F \in K^b$ and $w < v$ is a vertex of the same color as v, then $F \setminus \{v\} \cup \{w\} \in K^b$.

We say that the order $<$ on $V = V_1 \uplus \cdots \uplus V_d$ used to compute K^b is $(l, l, \ldots, l)$-*admissible* if the least l vertices from each colorset form an initial segment of $<$.

Recall that by Euler's formula, a planar graph with $n \geq 3$ vertices has at most $3n-6$ edges, and equality holds for the 1-dimensional skeleton of any triangulated 2-sphere. Conjecture 1.4 posits an analogous statement for 2-dimensional complexes embeddable in $\mathbb{R}^4$. What happens in higher dimensions?

For a simplicial complex K, let $f_i(K)$ be the number of i-dimensional faces (*i-faces*) of K, and let $f(K)$ be the *f-vector* of K, namely, $f(K) := (f_{-1}(K), f_0(K), \ldots, f_{\dim(K)}(K))$. It follows from the Dehn-Sommerville relations [23] and the generalized lower bound theorem [39] that if K is a $2d$-dimensional simplicial sphere that is the boundary of a polytope, then $f_d(K)$ is linear in $f_{d-1}(K)$. Is it true that for *any* d-dimensional complex K topologically embeddable in the $2d$-sphere, $f_d(K)$ is at most linear in $f_{d-1}(K)$? (That is, is there some constant $c(d)$, depending only on d, such that $f_d(K)/f_{d-1}(K) \leq c(d)$ for all relevant K?) We consider this question in the next section; we refer there to such inequality as *Euler-type upper bound* inequality.

8. Balanced complexes, Euler-type upper bounds, and the Kalai–Sarkaria conjecture

In this section we discuss a balanced analog of (a part of) the Kalai–Sarkaria conjecture, potential applications of this conjecture and possible approaches to attack it. Our starting point is the following conjecture of Kalai and Sarkaria [22] that implies McMullen's g-conjecture for simplicial spheres [27]. We let $C(d,n)$ denote the cyclic d-polytope with n vertices, $\partial(C(d,n))$ stands for the boundary complex of $C(d,n)$, and $\mathbb{S}^d$ denotes the d-dimensional sphere. Finally, for a simplicial complex K, K^s denotes the symmetric algebraic shifting of K.

Conjecture 8.1. *Let L be a simplicial complex with n vertices. If L is topologically embeddable in $\mathbb{S}^{d-1}$, then $L^s \subseteq (\partial(C(d,n)))^s$. In particular, if K is a d-dimensional complex embeddable in $\mathbb{S}^{2d}$, then K^s does not contain the Flores complex $\binom{[2d+3]}{\leq d+1}$.*

We posit the following bipartite analog of the "in particular" part:

Conjecture 8.2. *Let K be a d-dimensional balanced complex that is topologically embeddable in $\mathbb{S}^{2d}$, and let $<$ be a $(2,2,\dots,2)$-admissible order. Then $K^{b,<}$ does not contain the van Kampen complex $[3]^{*(d+1)}$, i.e., the $(d+1)$-fold join of 3 points.*[4]

As with the Kalai–Sarkaria conjecture, Conjecture 8.2 is known so far only for $d = 0,1$: the case $d=0$ is obvious and the case $d=1$ is Theorem 4.1. We observe that Conjecture 8.2 implies a weaker version of Conjecture 8.1 concerning Euler-type upper bound inequalities (see [15]) for *all* simplicial complexes (cf. Conjecture 1.4):

Proposition 8.3. *If Conjecture 8.2 is true, then for every nonnegative integer d the following holds:*

(i) *If Γ is a d-dimensional balanced complex that embeds in $\mathbb{S}^{2d}$, then*
$$f_d(\Gamma) \leq 2f_{d-1}(\Gamma).$$

(ii) *There exists a constant $c(d)$ such that for an arbitrary d-dimensional simplicial complex K that embeds in $\mathbb{S}^{2d}$, $f_d(K) \leq c(d) f_{d-1}(K)$.*

Proof. (i) It follows from Conjecture 8.2 that for any facet F in Γ^b there must be a colorset V_i such that F contains one of the two *minimal* elements of V_i. Since the total order $<$ on V is $(2,2,\dots,2)$-admissible, we conclude that the map $F \mapsto F\setminus\{\min_<(F)\}$ from the set of facets of Γ^b to the set of $(d-1)$-faces of Γ^b, is at most $2:1$. The fact that balanced shifting preserves f-vectors then yields $f_d(\Gamma) \leq 2f_{d-1}(\Gamma)$.

(ii) In a random coloring of the vertices of K by $d+1$ colors, the probability that a given facet is colorful (i.e., contains a vertex of each color) is $\frac{(d+1)!}{(d+1)^{d+1}}$. Thus, there is a coloring with at least $\frac{(d+1)!}{(d+1)^{d+1}} f_d(K)$ colorful facets; denote by L the balanced subcomplex of K spanned by these facets. Then by part (i),
$$f_d(K) \leq \frac{(d+1)^{d+1}}{(d+1)!} f_d(L) \leq \frac{(d+1)^{d+1}}{(d+1)!} 2f_{d-1}(L) \leq \frac{2(d+1)^{d+1}}{(d+1)!} f_{d-1}(K).$$
Hence, taking $c(d) = 2(d+1)^{d+1}/(d+1)!$ completes the proof. □

We remark that Conjecture 8.1, if true, would imply that $c(d) = d+2$, while from Conjecture 8.2 we only derived the weaker estimate of $c(d) = 2(d+1)^{d+1}/(d+1)!$.

We next show that the above Euler-type inequality implies a weaker version of Conjecture 8.2, so "up to constants" they are equivalent; more precisely:

[4]The statements of Conjectures 8.1, 1.7, and 8.2 are also conjectured to hold for the case of exterior shifting (balanced exterior shifting, resp.). In fact, in an unpublished work, Nevo established the exterior shifting counterpart of Proposition 1.6.

Proposition 8.4. *Assume there is a constant $c(d)$ such that for every d-dimensional balanced complex K embeddable in $\mathbb{S}^{2d}$, $f_d(K) \leq c(d) f_{d-1}(K)$. Let $C(d) = (d{+}1)c(d)$. Then for every d-dimensional balanced complex K embeddable in $\mathbb{S}^{2d}$ and a $(C(d), \ldots, C(d))$-admissible order $<$, $K^{b,<}$ does not contain $[C(d)+1]^{*(d+1)}$.*

Proof. Our assumption that $f_d(K) \leq c(d) f_{d-1}(K)$ implies that there is a ridge of K that is contained in at most $(d+1)c(d)$ facets of K. Now use the high-dimensional Deletion Lemma (see Lemma 8.9 below) and induction. □

We now turn to rephrasing Conjecture 8.2 in terms of embeddability of K^b, a formulation that is *not* available for Conjecture 8.1: indeed, while shifted graphs not containing K_5 may be nonplanar (for instance, G^s where G is the graph of the octahedron), *balanced-shifted* bipartite graphs not containing $K_{3,3}$ are necessarily planar. This statement extends to higher dimensions, as the following proposition shows.

Proposition 8.5. *Let K be a d-dimensional balanced-shifted simplicial complex not containing $[3]^{*(d+1)}$ as a subcomplex. Then K is PL embeddable in $\mathbb{S}^{2d}$.*

Proof. Among all d-dimensional balanced-shifted simplicial complexes on the same vertex set as K, let $\Gamma(d)$ be the maximal complex not containing $[3]^{*(d+1)}$. In other words, the facets of $\Gamma(d)$ are all the colorful $(d+1)$-subsets of V that contain one of the least two vertices of some color. We need to show that $\Gamma(d)$ is PL embeddable in $\mathbb{S}^{2d}$.

For $d = 0$ this is clear, as $\Gamma(d)$ consists of two points. For $d = 1$, this is also easy: in the plane, draw a square with vertices $1, 1', 2, 2'$; embed the vertices $3, 4, \ldots, n$ in the open segment connecting 1 and 2, and the vertices $3', 4', \ldots, m'$ in the parts of the straight line through $1'$ and $2'$ that lie outside of the square; now draw as straight segments the edges ij' where at least one of $i, j \leq 2$.

We show by induction on d how to PL embed $\Gamma(d)$ in $\mathbb{R}^{2d}$ for $d > 1$. Consider the first d (out of $d+1$) colorsets of $\Gamma(d)$, and two subcomplexes of $\Gamma(d)$ on these colors: $\Gamma(d-1)$ and $\Gamma(d)_{[d]}$. (Note that $\Gamma(d-1) \subseteq \Gamma(d)_{[d]}$.) Assume that $\Gamma(d-1)$ is PL embedded in $\mathbb{R}^{2d-2} \times \{0\} \times \{0\}$. As $\dim \Gamma(d-1) = \dim \Gamma(d)_{[d]} = d-1$, we can extend this embedding to a PL map from $\Gamma(d)_{[d]}$ into $\mathbb{R}^{2d-2} \times \{0\} \times \{0\}$ in such a way that (i) the only intersections occur between pairs of facets that involve at least one of the "added" faces (i.e., faces of $\Gamma(d)_{[d]}$ that do not belong to $\Gamma(d-1)$), (ii) they occur at interior points, and (iii) there are finitely many such points. Now resolve these intersections by pulling the added $(d-1)$-faces, one by one, into the *negative* side of the last coordinate (keeping the coordinate before last equal to zero). Figure 1 illustrates the case of $d = 2$, $n = 4$, $m' = 3'$.

Next, place the first and second vertices of color $d+1$ at $\pm v$, where v is the unit vector with the coordinate before last equal to 1, and consider two geometric cones over the above embedding of $\Gamma(d)_{[d]}$: one with apex v and another one with apex $-v$. The union of these cones provides an embedding of the suspension of $\Gamma(d)_{[d]}$, $\Sigma(\Gamma(d)_{[d]})$, and this embedding is such that the last coordinate is always nonpositive.

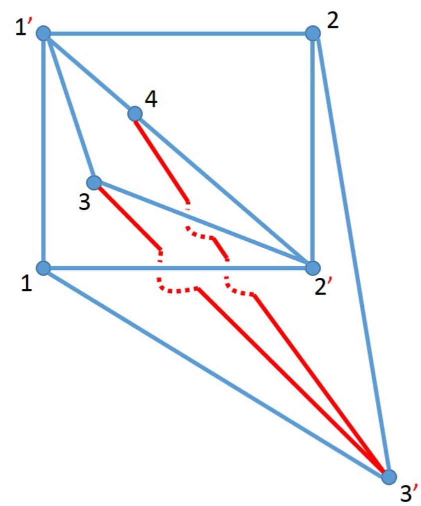

FIGURE 1. The first step of the case $d = 2$ (with $n = 4, m = 3$). The solid lines are the edges of $\Gamma(1)$; the partially dashed lines are the edges of $\Gamma(2)_{[2]} - \Gamma(1)$.

Finally, place the remaining vertices (i.e., vertices number $3, 4, \ldots$) of color $d + 1$ at distinct points on the open arc $\{(0, \cdots, 0, t, s) : \ t^2 + s^2 = 1, \ s, t > 0\} \subset \mathbb{R}^{2d-2} \times \mathbb{R}^2$, and for each of those points, construct a geometric cone over $\Gamma(d-1)$ with that point as the apex. These cones lie in distinct half hyperplanes with a common boundary, namely $\mathbb{R}^{2d-2} \times \{0\} \times \{0\}$, and hence this union of cones is embedded. All the new points added in this step have a *positive* last coordinate, and thus are disjoint from the embedding of $\Sigma(\Gamma(d)_{[d]})$. Together with that embedding of $\Sigma(\Gamma(d)_{[d]})$, they form an embedding of $\Gamma(d)$ into $\mathbb{R}^{2d}$. $\square$

Combining Proposition 8.5 with the well-known fact that the complex $[3]^{*(d+1)}$ is not PL embeddable in $\mathbb{S}^{2d}$ [40, 11], we obtain that Conjecture 8.2 is equivalent to the following:

Conjecture 8.6. *If K is a d-dimensional balanced complex that is topologically embeddable in $\mathbb{S}^{2d}$, and $<$ is a $(2, \ldots, 2)$-admissible order, then $K^{b,<}$ is PL embeddable in $\mathbb{S}^{2d}$.*

Let $\mathfrak{o}(K)$ denote the van Kampen obstruction to PL embeddability of a d-dimensional complex K in $\mathbb{S}^{2d}$, computed with coefficients in $\mathbb{Z}$. (One may also use other coefficients, e.g., $\mathbb{Z}/2\mathbb{Z}$). Recall that if K is PL embeddable in $\mathbb{S}^{2d}$, then $\mathfrak{o}(K) = 0$ (and the converse also holds provided $d \neq 2$), see [35, 47, 12]. As $\mathfrak{o}([3]^{*(d+1)}) \neq 0$ (even with $\mathbb{Z}_2$ coefficients) and as, according to [7], for $d \geq 3$ the topological embeddability of a d-dimensional complex K in $\mathbb{S}^{2d}$ is equivalent to the PL embeddability, we obtain that for $d \geq 3$, the following conjecture implies Conjecture 8.6, even when considered with $\mathbb{Z}_2$ coefficients.

Conjecture 8.7. *Let K be a d-dimensional balanced complex. If $\mathfrak{o}(K) = 0$, then $\mathfrak{o}(K^b) = 0$.*

We are now in a position to introduce a balanced rigidity matrix corresponding to Conjecture 8.2. As with bipartite graphs, for a d-dimensional balanced complex K and a fixed integer l, assign to each vertex $v \in K$ a generic l-dimensional real vector θ_v, and define the following facet-ridge matrix $M(K, l)$: the rows of $M(K, l)$ correspond to the facets F of K, the columns of $M(K, l)$ come in l-tuples with each l-tuple corresponding to a ridge G of K; the $1 \times l$ block of $M(K)_{F,G}$ is (0) if $G \nsubseteq F$ and θ_{F-G} otherwise. (Thus, if K is 1-dimensional then $M(K, l) = R^{(l,l)}(K)$.) Arguing as in Lemma 2.3 (and Proposition 3.3), we obtain:

Lemma 8.8. *For an $(l, \ldots, l)$-admissible order $<$, the complex $K^{b,<}$ does not contain $[l + 1]^{*(d+1)}$ as a subcomplex if and only if the rows of the matrix $M(K, l)$ are linearly independent.*

The following is a high-dimensional analog of the Deletion Lemma (Lemma 3.7).

Lemma 8.9. *Let K be a d-dimensional balanced complex, $<$ an $(l, \ldots, l)$-admissible order, G a ridge of K contained in at most l facets of K, and $L = \mathrm{ast}_K(G)$. If $[l + 1]^{*(d+1)} \nsubseteq L^{b,<}$ then $[l + 1]^{*(d+1)} \nsubseteq K^{b,<}$.*

Proof. Let $s \leq l$ denote the number of facets of K that contain G. The matrix $M(K, l)$ is obtained from $M(L, l)$ by adjoining l columns corresponding to G and s rows corresponding to the facets containing G. These new l columns consist of zeros followed by a generic $s \times l$ block (at the intersection with the new s rows). Thus $\mathrm{rank}\, M(K, l) \geq \mathrm{rank}\, M(L, l) + s$. Since by our assumption on L, $\mathrm{rank}\, M(L, l)$ equals the number of rows of $M(L, l)$, the quantity $\mathrm{rank}\, M(L, l) + s$ coincides with the number of rows of $M(K, l)$. Hence, the rows of $M(K, l)$ are linearly independent. □

In the rest of the section, we gather some evidence in favor of Conjecture 8.2. Specifically, we consider certain basic constructions on balanced simplicial complexes and their effect on balanced shifting. We start with the join operation. All balanced shiftings in the rest of this section are computed w.r.t. $(2, \ldots, 2)$-admissible orders.

Lemma 8.10. *Let K be a k-dimensional balanced complex embeddable in $\mathbb{S}^{2k}$, and let L be any l-dimensional balanced complex. Then $K * L$ is a $(k+l+1)$-dimensional balanced complex embeddable in the $2(k + l + 1)$-sphere. Moreover, if K satisfies the conclusion of Conjecture 8.2, then so does $K * L$.*

Proof. As any l-dimensional simplicial complex embeds in the $(2l + 1)$-sphere, our assumption on K implies that $K * L$ embeds in the $2(k+l+1)$-sphere. Assume that K^b and L^b are computed w.r.t. linear $(2, \ldots, 2)$-admissible orders $<_K$ and $<_L$, respectively, and that $(K * L)^b$ is computed w.r.t. a linear order $<$ that extends the partial order $<_K \uplus <_L$. It then follows from the definition of the balanced shifting that $(K * L)^b = K^b * L^b$. Thus, if K^b does not contain $[3]^{*(k+1)}$, then $K^b * L^b$ does not contain $[3]^{*(k+l+2)}$ as it does not even contain its subcomplex $[3]^{*(k+1)}$ on the first $k + 1$ colors. □

Next we consider the effect of certain subdivisions. To do so, for a balanced complex L, we use the balanced rigidity matrix $M(L) := M(L,2)$.

Let K be a pure balanced d-dimensional complex and σ a face of K that is not a vertex. Let S be any pure balanced complex of the same dimension as σ and assume that S has a missing facet X. Identify the vertices of this missing facet with the correspondingly colored vertices of σ and define

$$K' = \mathrm{ast}_K(\sigma) \cup (S * \mathrm{lk}_K(\sigma)).$$

In other words, K' is obtained from K by removing the star of σ, $\overline{\sigma} * \mathrm{lk}_K(\sigma)$, and replacing it with $S * \mathrm{lk}_K(\sigma)$. Then K' is a pure balanced d-dimensional complex. Further, if S is a ball whose boundary coincides with that of X (i.e, S is obtained from a balanced sphere by removing one facet, X), then K' is homeomorphic to K.

Proposition 8.11. *If $(\mathrm{ast}_K(\sigma))^b$ does not contain $[3]^{*(d+1)}$ and $(\mathrm{lk}_K(\sigma))^b$ does not contain $[3]^{*(d-|\sigma|+1)}$, then $(K')^b$ does not contain $[3]^{*(d+1)}$.*

Proof. Denote the facets of S by $H_1, \dots, H_m$. The facets of K' then fall in two categories: (a) the facets of $\mathrm{ast}_K(\sigma)$, and (b) for each $i = 1, \dots, m$, the facets of the form $G \cup H_i$ where G is a facet of $\mathrm{lk}_K(\sigma)$; we denote this i-th set of facets by $\mathcal{F}_i$. Similarly, the ridges of K' are of the following types: (a) the ridges of $\mathrm{ast}_K(\sigma)$, (b) for each $i = 1, \dots, m$, the ridges of the form $R \cup H_i$ where R is a ridge of $\mathrm{lk}_K(\sigma)$ — we denote this i-th set of ridges by $\mathcal{R}_i$, and (c) all remaining ridges. In the following we will ignore the ridges of type (c); specifically, we will show that the restriction of $M(K')$ to the columns of the ridges of types (a) and (b) already has independent rows.

Consider the balanced rigidity matrix $M(K')$. Its restriction to facets/ridges of $\mathrm{ast}_K(\sigma)$ coincides with the restriction of $M(K)$ to the same rows and columns, and thus, by our assumption on $\mathrm{ast}_K(\sigma)$, has rank $f_d(\mathrm{ast}_K(\sigma))$ (see Lemma 8.8). For each $i = 1, \dots, m$, the restriction of $M(K')$ to the columns labeled by the ridges from $\mathcal{R}_i$ consists of the block $M(\mathrm{lk}_K(\sigma))$ positioned at the intersection with the rows labeled by the elements of $\mathcal{F}_i$, and zeros everywhere else. By our assumption on the link, the rank of such a block equals the number of facets of the link. As all these blocks have pairwise disjoint sets of columns and rows, it follows that

$$\begin{aligned} \mathrm{rank}(M(K')) &\geq \mathrm{rank}(M(\mathrm{ast}_K(\sigma)) + m \cdot \mathrm{rank}(M(\mathrm{lk}_K(\sigma)) \\ &= f_d(\mathrm{ast}_K(\sigma)) + f_{|\sigma|-1}(S) \cdot f_{d-|\sigma|}(\mathrm{lk}_K(\sigma)) = f_d(K'). \end{aligned}$$

Thus the above inequality is, in fact, equality, and $(K')^b$ does not contain $[3]^{*(d+1)}$. $\square$

Finally, we show that if S is obtained from a $(|\sigma|-1)$-dimensional balanced pseudomanifold by removing one facet, X, then the condition on the link in Proposition 8.11 can be dropped. More generally:

Proposition 8.12. *Let K be a pure balanced d-dimensional complex and σ a face of K that is not a vertex. Let S be a pure $(|\sigma|-1)$-dimensional balanced simplicial complex with a missing facet X and such that each ridge of S is in at most two facets. Let*

$K' = \mathrm{ast}_K(\sigma) \cup (S * \mathrm{lk}_K(\sigma))$. *If* $(\mathrm{ast}_K(\sigma))^b$ *does not contain* $[3]^{*(d+1)}$, *then* $(K')^b$ *does not contain* $[3]^{*(d+1)}$.

Proof. Delete from the matrix $M(S)$ all columns corresponding to the ridges that are subsets of X; denote the resulting matrix by $M^*(S)$. Note that $M^*(S)$ has no zero rows: this is because every facet of S has at least one ridge that is not a subset of X. Moreover, every ridge is in at most two facets, and so the rows of $M^*(S)$ are linearly independent (the same argument as in the proof of Lemma 8.9 applies.)

Now let the facets of $\mathrm{lk}_K(\sigma)$ be $H_1, \dots, H_k$. The set of facets of K' consists of (a) the facets of $\mathrm{ast}_K(\sigma)$, and (b) for each $i = 1, \dots, k$, the facets of the form $G \cup H_i$, where G is a facet of S; denote this i-th set of facets by $\mathcal{F}_i$. The set of ridges of K' consists of (a) the ridges of $\mathrm{ast}_K(\sigma)$, (b) for each $i = 1, \dots, k$, the ridges of the form $R \cup H_i$ where R is a ridge of S not contained in X — denote this i-th set of ridges by $\mathcal{R}_i$, (c) all remaining ridges, which we will ignore.

We again consider the balanced rigidity matrix of K'. As in Proposition 8.11, the restriction of $M(K')$ to facets/ridges of $\mathrm{ast}_K(\sigma)$ has rank $f_d(\mathrm{ast}_K(\sigma))$. For each $i = 1, \dots, k$, the restriction of $M(K')$ to the columns labeled by the ridges from $\mathcal{R}_i$ consists of the block $M^*(S)$ (positioned at the intersection with the rows labeled by the elements of $\mathcal{F}_i$) and zeros everywhere else. By the first paragraph of this proof, the rank of such a block equals the number of facets of S. As all these blocks have pairwise disjoint sets of columns and rows, we obtain that

$$\begin{aligned} \mathrm{rank}(M(K')) &\geq \mathrm{rank}(\mathrm{ast}_K(\sigma)) + k \cdot \mathrm{rank}(M^*(S)) \\ &= f_d(\mathrm{ast}_K(\sigma)) + f_{d-|\sigma|}(\mathrm{lk}_K(\sigma)) \cdot f_{|\sigma|-1}(S) = f_d(K'). \end{aligned}$$

The result follows. □

We conclude with a conjecture on linklessly embedable complexes. A high-dimensional analog of Sachs' result [34] on linkless embeddability is due to Skopenkov [37, Lemma 1]. It asserts that $[4]^{*(d+1)}$ is not linklessly embeddable in $\mathbb{R}^{2d+1}$. This theorem leads us to pose the following generalization of Conjecture 4.5, analogous to Conjecture 8.2.

Conjecture 8.13. *Let* K *be a* d*-dimensional balanced simplicial complex that is linklessly embeddable in* $\mathbb{R}^{2d+1}$ *and let* $<$ *be a* $(3, \dots, 3)$*-admissible order. Then* $K^{b,<}$ *does not contain* $[4]^{*(d+1)}$.

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References

[1] L. Asimow and B. Roth. The rigidity of graphs. *Trans. Amer. Math. Soc.*, 245:279–289, 1978.
[2] L. Asimow and B. Roth. The rigidity of graphs. II. *J. Math. Anal. Appl.*, 68(1):171–190, 1979.

[3] E. Babson and C. Chan. Counting faces of cubical spheres modulo two. *Discrete Math.*, 212(3):169–183, 2000. Combinatorics and applications (Tianjin, 1996).

[4] E. Babson and I. Novik. Face numbers and nongeneric initial ideals. *Electron. J. Combin.*, 11(2):Research Paper 25, 23 pp. (electronic), 2004/06.

[5] E. Babson, I. Novik, and R. Thomas. Reverse lexicographic and lexicographic shifting. *J. Algebraic Combin.*, 23(2):107–123, 2006.

[6] G. Blind and R. Blind. Gaps in the numbers of vertices of cubical polytopes. I. *Discrete Comput. Geom.*, 11(3):351–356, 1994.

[7] J. L. Bryant. Approximating embeddings of polyhedra in codimension three. *Trans. Amer. Math. Soc.*, 170:85–95, 1972.

[8] M. Chudnovsky, G. Kalai, E. Nevo, I. Novik, and P. Seymour. Bipartite minors. Preprint, arXiv:1312.0210.

[9] R. Connelly. A counterexample to the rigidity conjecture for polyhedra. *Inst. Hautes Études Sci. Publ. Math.*, (47):333–338, 1977.

[10] R. Connelly. Rigidity. In *Handbook of convex geometry, Vol. A, B*, pages 223–271. North-Holland, Amsterdam, 1993.

[11] A. Flores. Über n-dimensionale komplexe die im $\mathbb{R}^{2n+1}$ absolut selbstverschlungen sind. *Ergeb. Math. Kolloq.*, 6:4–7, 1933/4.

[12] M. H. Freedman, V. S. Krushkal, and P. Teichner. van Kampen's embedding obstruction is incomplete for 2-complexes in $\mathbf{R}^4$. *Math. Res. Lett.*, 1(2):167–176, 1994.

[13] H. Gluck. Almost all simply connected closed surfaces are rigid. In *Geometric topology (Proc. Conf., Park City, Utah, 1974)*, pages 225–239. Lecture Notes in Math., Vol. 438. Springer, Berlin, 1975.

[14] J. E. Goodman and H. Onishi. Even triangulations of S^3 and the coloring of graphs. *Trans. Amer. Math. Soc.*, 246:501–510, 1978.

[15] B. Grünbaum. Higher-dimensional analogs of the four-color problem and some inequalities for simplicial complexes. *J. Combinatorial Theory*, 8:147–153, 1970.

[16] W. Jockusch. The lower and upper bound problems for cubical polytopes. *Discrete Comput. Geom.*, 9(2):159–163, 1993.

[17] M. Joswig. Projectivities in simplicial complexes and colorings of simple polytopes. *Math. Z.*, 240:243–259, 2002.

[18] G. Kalai. Characterization of f-vectors of families of convex sets in $\mathbf{R}^d$. I. Necessity of Eckhoff's conditions. *Israel J. Math.*, 48(2-3):175–195, 1984.

[19] G. Kalai. Hyperconnectivity of graphs. *Graphs Combin.*, 1(1):65–79, 1985.

[20] G. Kalai. Rigidity and the lower bound theorem. I. *Invent. Math.*, 88(1):125–151, 1987.

[21] G. Kalai. The diameter of graphs of convex polytopes and f-vector theory. In *Applied geometry and discrete mathematics*, volume 4 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 387–411. Amer. Math. Soc., Providence, RI, 1991.

[22] G. Kalai. Algebraic shifting. In *Computational commutative algebra and combinatorics (Osaka, 1999)*, volume 33 of *Adv. Stud. Pure Math.*, pages 121–163. Math. Soc. Japan, Tokyo, 2002.

[23] V. Klee. A combinatorial analogue of Poincaré's duality theorem. *Canad. J. Math.*, 16:517–531, 1964.

[24] G. Laman. On graphs and rigidity of plane skeletal structures. *J. Engrg. Math.*, 4:331–340, 1970.

[25] C. W. Lee. Generalized stress and motions. In *Polytopes: abstract, convex and computational (Scarborough, ON, 1993)*, volume 440 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 249–271. Kluwer Acad. Publ., Dordrecht, 1994.

[26] W. Mader. Homomorphiesätze für Graphen. *Math. Ann.*, 178:154–168, 1968.

[27] P. McMullen. The numbers of faces of simplicial polytopes. *Israel J. Math.*, 9:559–570, 1971.

[28] J. R. Munkres. *Elements of algebraic topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[29] E. Nevo. *Algebraic Shifting and f-Vector Theory*. PhD thesis, Hebrew University, Jerusalem, 2007.
[30] E. Nevo. Higher minors and Van Kampen's obstruction. *Math. Scand.*, 101(2):161–176, 2007.
[31] E. Nevo. On embeddability and stresses of graphs. *Combinatorica*, 27(4):465–472, 2007.
[32] I. Pak. *Lectures on discrete and polyhedral geometry*. Book, in preparation, available at http://www.math.ucla.edu/~pak/book.htm.
[33] N. Robertson, P. Seymour, and R. Thomas. Sachs' linkless embedding conjecture. *J. Combin. Theory Ser. B*, 64(2):185–227, 1995.
[34] H. Sachs. On a spatial analogue of Kuratowski's theorem on planar graphs—an open problem. In *Graph theory (Łagów, 1981)*, volume 1018 of *Lecture Notes in Math.*, pages 230–241. Springer, Berlin, 1983.
[35] A. Shapiro. Obstructions to the imbedding of a complex in a euclidean space. I. The first obstruction. *Ann. of Math. (2)*, 66:256–269, 1957.
[36] A. Singer and M. Cucuringu. Uniqueness of low-rank matrix completion by rigidity theory. *SIAM J. Matrix Anal. Appl.*, 31(4):1621–1641, 2009/10.
[37] M. Skopenkov. Embedding products of graphs into Euclidean spaces. *Fund. Math.*, 179(3):191–198, 2003.
[38] R. P. Stanley. Balanced Cohen-Macaulay complexes. *Trans. Amer. Math. Soc.*, 249(1):139–157, 1979.
[39] R. P. Stanley. The number of faces of a simplicial convex polytope. *Adv. in Math.*, 35(3):236–238, 1980.
[40] E. R. van Kampen. Komplexe in euklidischen räumen. *Abh. Math. Sem.*, 9:72–78, 1932.
[41] K. Wagner. Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.*, 114(1):570–590, 1937.
[42] U. Wagner. Minors in random and expanding hypergraphs. In *Computational geometry (SCG'11)*, pages 351–360. ACM, New York, 2011.
[43] W. Whiteley. Cones, infinity and 1-story buildings. *Structural Topology*, (8):53–70, 1983. With a French translation.
[44] W. Whiteley. A matroid on hypergraphs, with applications in scene analysis and geometry. *Discrete Comput. Geom.*, 4(1):75–95, 1989.
[45] W. Whiteley. Vertex splitting in isostatic frameworks. *Struc. Top.*, 16:23–30, 1989.
[46] W. Whiteley. Some matroids from discrete applied geometry. In *Matroid theory (Seattle, WA, 1995)*, volume 197 of *Contemp. Math.*, pages 171–311. Amer. Math. Soc., Providence, RI, 1996.
[47] W.-t. Wu. *A theory of imbedding, immersion, and isotopy of polytopes in a euclidean space*. Science Press, Peking, 1965.

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