The Discrete Harmonic Cohomology Module on Networks

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Contents

1 Graphs and Harmonic Functions

1.1 Graphs with Boundary

A **graph** G consists of a vertex set V, an edge set E, and a one-to-one assignment of endpoints (an unordered pair of distinct vertices) to each edge. This definition rules out loops from a vertex to itself and multiple edges between two vertices; additionally, we assume in this paper that all graphs are finite. We use the notation $v_1 \sim e \sim v_2$ if v_1 and v_2 are the endpoints of e, and say v_1 and v_2 are **incident** to the edge e. We also write $v_1 \sim v_2$ if there is an edge between v_1 and v_2, and we may denote this edge as $v_1 v_2$. The **valence** of a vertex is the number of vertices that it is incident to.

In this paper, we consider by default only **connected** graphs: those such that, for any vertices v and w, there is a sequence $v, v_1, \ldots, v_n, w$ in which each pair of consecutive vertices is joined by an edge.

A morphism of graphs $f : (V, E) \to (V', E')$ is a pair of functions $f_V : V \to V'$ and $f_E : E \to E'$ such that if the endpoints of e are v_1 and v_2, the endpoints of $f_E(e)$ are $f_V(v_1)$ and $f_V(v_2)$.

A **graph with boundary**, **boundary graph**, or **bgraph** is a graph $G = (V, E)$ together with a partition $V = \partial V \cup \operatorname{int} V$ into **boundary vertices** ∂V and **interior vertices** $\operatorname{int} V$, with ∂V nonempty. A **morphism of boundary graphs** is a graph morphism f with two additional properties [2][p. 4]:

- f sends interior vertices to interior vertices.
- For an interior vertex v, if the map is restricted to a map on the neighbors of v, the preimage of every neighbor of $f(v)$ has the same size n, $n \geq 1$. (In particular, this implies that the map must be surjective onto the image of the neighbors of the interior vertex.)

A **sub-boundary-graph** or **sub-bgraph** of a boundary graph is an ordinary subgraph such that the inclusion into the supergraph is a bgraph morphism. In particular, since the inclusion map must be one-to-one, it must be a bijection when restricted to the neighbors of an interior vertex, by the second point above.

Let $G = (\partial V \cup \operatorname{int} V, E)$ be a graph with boundary and let R be a commutative ring with 1. (This theory can be extended with little difficulty to non-commutative rings, but we do not consider any in this paper.) Then G can be given the structure of a **network** with a function $\gamma : E \to R - \{0\}$ that assigns a **conductance** to each edge. In many algebraic contexts, we take the conductances to be units, and that is assumed here unless stated otherwise.

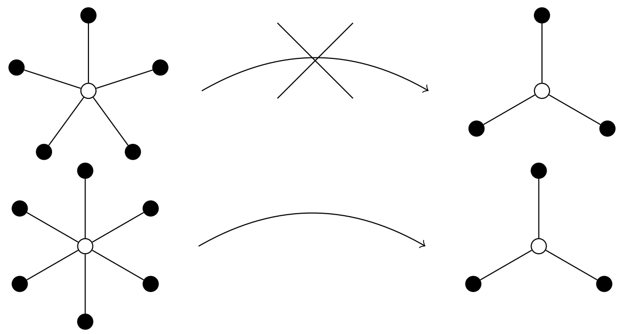


Figure 1: There is no bgraph morphism from the 5-star to the 3-star, since it would have to be n-to-1 from 5 neighbors of the first interior vertex to 3 neighbors of the other. However, there is a bgraph morphism from the 6-star to the 3-star, which is 2-to-1 in the neighborhood of the interior vertex.

We refer to the network $\Gamma = (\partial V \cup \operatorname{int} V, E, \gamma)$. If no conductance function is specified on a network, it is assumed by default that all the conductances are 1. A **network morphism** is then a morphism of the underlying networks that preserves conductances.

There are certain ways of reducing bgraphs to sub-bgraphs that we will want to consider [2, p.25]. A **boundary edge** is an edge between two boundary vertices. A **boundary spike** is an edge which has an interior vertex as one endpoint and a boundary vertex of valence 1 as the other. An **isolated boundary vertex** is a boundary vertex that is adjacent to no other vertices. A **boundary cutpoint** is a boundary vertex which, if removed, disconnects the graph. Often we will want to remove these types of edges/vertices from graphs.

Deleting a boundary edge or an isolated boundary vertex means creating a new bgraph by removing that edge or vertex. Deleting (or **contracting**) a boundary spike means creating a new bgraph by removing the edge and boundary vertex associated with the spike and changing the interior vertex into a boundary vertex.

Splitting a boundary cutpoint is a slightly more complicated procedure. First, removing the vertex (and its incident edges) creates a disconnected subgraph. To each connected component, reattach a distinct copy of the deleted boundary vertex with the edges that originally connected it to that component.

A graph is **layerable** if it can be reduced to the empty graph (the graph with no vertices) by a series of deletions of boundary edges, boundary spikes, and isolated boundary vertices. It is **quasi-layerable** if it can be reduced to the empty graph using these operations along with splitting of boundary cutpoints.[1]

If one of the connected components obtained by deleting a boundary cutpoint is a single interior vertex, the interior vertex must have originally had degree 1. We refer to this special case as an **interior spike**. Interior spikes are irrelevant

[1] The algebraic considerations in this paper suggest that quasi-layerability may be a more natural definition, but for now, we reserve "layerability" to refer to Jekel's original definition.

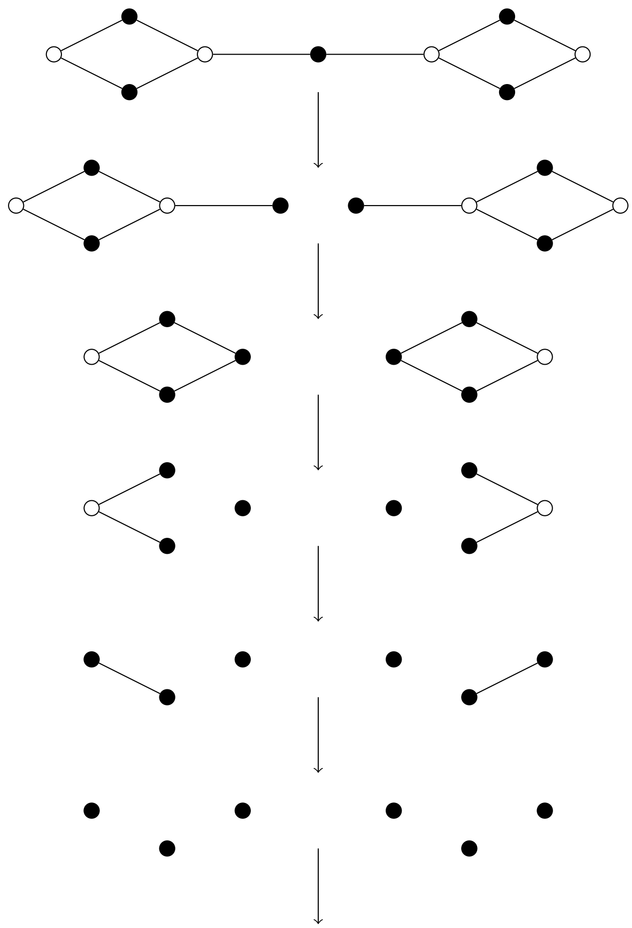


Figure 2: Reducing a quasi-layerable graph by splitting a boundary cutpoint, contracting boundary spikes, deleting boundary edges, contracting further spikes, deleting further edges, and then deleting the isolated vertices. From the second step on, the graphs are also layerable.

to the electrical properties of the network, so for most of this paper we will assume graphs do not have them.

1.2 Harmonic Functions

When the ring $R = \mathbb{R}$ and the conductances are restricted to positive values, this structure resembles an electrical network. With this in mind, for some general left R-module M, we can define a **potential** function on the network Γ to be any function $u : V \to M$. Then for a specific potential, the **current** along an edge e from vertex v_1 to v_2 is given by $\gamma(e)(u(v_1)-u(v_2))$. (This linear relationship is derived from Ohm's law, where our conductance is the reciprocal of resistance.)

In investigating electrical networks, we consider assignments of potentials that satisfy Kirchhoff's Current Law: the net current leaving an interior vertex along all edges is 0. Stated using our notation,

$$\sum_{e,w:v\sim e\sim w} \gamma(e)(u(v) - u(w)) = 0 \quad \forall v \in .$$

A function u that satisfies this property is called **γ-harmonic**, or just **harmonic**. We denote the set of harmonic functions on the network Γ with values in M by $\mathcal{U}_{\Gamma,M}$.

Note that $\mathcal{U}_{\Gamma,M} \subseteq M^V$ has the structure of an R-module. The operator $\sum_{e,w:v\sim e\sim w} \gamma(e)(u(v) - u(w))$ which takes a function u to the resulting net current at a vertex v is a homomorphism of R-modules, and so its kernel $\mathcal{U}_{\Gamma,M}$ is a submodule. (In the case where R is not commutative, it can only be viewed as a $\mathbb{Z}$-module, but the theory is mostly the same.)

1.3 The Dirichlet Problem and The Kirchhoff Matrix

The (discrete) **Dirichlet problem** asks: given arbitrary potential values at the boundary vertices, is there a harmonic function which takes those values, and is it unique? The question can be simplified significantly with the use of matrices.

As mentioned above, the relationship between the potential at a vertex and the net current there is linear, and it can be expressed simply as a matrix obtained from the values of γ.

Given a network $\Gamma = (\partial V \cup \operatorname{int} V, E, \gamma)$, label the vertices $V = \{v_1, v_2, \ldots, v_n\}$ and define the $n \times n$ Kirchhoff matrix $K = \{k_{ij}\}$ by

$$k_{ij} = \begin{cases} \sum_{v_i \sim e} \gamma(e) & i = j \\ -\gamma(e_{ij}) & v_i \text{ is adjacent to } v_j \text{ through the edge } e_{ij} \\ 0 & \text{otherwise} \end{cases}$$

This definition is such that, given a potential function u and the vector $\vec{u} = (u(v_1), u(v_2), \ldots, u(v_n))$, the resulting entry $(K\vec{u})_i$ gives the net current at vertex v_i.

It is standard to number the vertices in such a way that all of the boundary vertices come first. Additionally, K is symmetric, since its definition does not depend on the order of i and j. This allows K to be partitioned into a block matrix

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

where:

- A is a symmetric $|\partial V| \times |\partial V|$ matrix representing the conductances of edges between boundary vertices;
- B is a $|\partial V| \times |\operatorname{int} V|$ matrix representing the conductances of edges between boundary vertices and interior vertices;
- C is a symmetric $|\operatorname{int} V| \times |\operatorname{int} V|$ matrix representing the conductances of edges between interior vertices.

Then, partitioning $\vec{u}$ into $(\vec{u}_{\partial V}, \vec{u}_{\operatorname{int} V})$ in the same way, the condition of harmonicity can be restated using K:

$$u \text{ is harmonic iff } \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} \vec{u}_{\partial V} \\ \vec{u}_{\operatorname{int} V} \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix},$$

where $*$ can be any value. This is the same as requiring that $\begin{pmatrix} B^T & C \end{pmatrix} \vec{u} = 0$, so we get a helpfully succinct description of the harmonic functions:

Proposition 1.1. *$\mathcal{U}_{\Gamma,M} = \ker(B^T, C)_M$, where the matrix is interpreted as an R-module homomorphism $M^V \to M^{\operatorname{int} V}$.*

Returning to the Dirichlet problem, if we are given potential values $\vec{u}_{\partial V}$ on the boundary, then for the remaining interior values $\vec{u}_{\operatorname{int} V}$ to give a harmonic function requires that

$$B^T \vec{u}_{\partial V} + C\vec{u}_{\operatorname{int} V} = 0$$
$$C\vec{u}_{\operatorname{int} V} = -B^T \vec{u}_{\partial V}$$

The Dirichlet problem reduces to considering whether this equation can be solved for $\vec{u}_{\operatorname{int} V}$. This naturally depends on the module M, but in the specific case that the ring R is a field, the problem has a unique solution for any boundary values exactly when $\det C \neq 0$. In general, if $\det C = 0$, Γ is said to be **Dirichlet singular**.

1.4 $\mathcal{U}$ as a Functor

As was mentioned above, for an R-module M, $\mathcal{U}_{\Gamma,M}$ can also be interpreted as an R-module. Additionally, given some $u \in \mathcal{U}_{\Gamma,M}$ and an R-module homomorphism $\phi : M \to N$, we can consider the function $\phi_* u = \phi \circ u : V \to N$; then since

$$\sum_{e,w:v\sim e\sim w} \gamma(e)(\phi(u(v)) - \phi(u(w)) = \phi\left(\sum_{e,w:v\sim e\sim w} \gamma(e)(u(v) - u(w))\right),$$

this new function is harmonic. We then have a homomorphism $\phi_* : \mathcal{U}_{\Gamma,M} \to \mathcal{U}_{\Gamma,N}$. Additionally, for module homomorphisms $\phi : L \to M$ and $\psi : M \to N$, we have $(\psi \circ \phi)_* u = \psi \circ \phi \circ u = \psi_* \phi_* u$. As a result, for any fixed network Γ, $\mathcal{U}_{\Gamma,-}$ is a (covariant) functor **R-Mod** $\to$ **R-Mod**.

However, we can also ask what happens when we fix the module and vary the network.

Proposition 1.2. *For a fixed R-module M, $\mathcal{U}_{-,M}$ is a contravariant functor* ***Network*** $\to$ ***R-Mod****, where for a network morphism $f : \Gamma \to \Gamma'$ the image under the functor is the pullback $f^*(u) = u \circ f$.*

Proof. First, we must show that, for $u \in \mathcal{U}_{\Gamma',M}$, $u \circ f$ is harmonic on Γ. Consider an arbitrary interior vertex v of Γ; then consider the sum

$$\sum_{e,w:v\sim e\sim w} \gamma(e)(u(f(v)) - u(f(w)))$$

Since v is an interior vertex of Γ, $f(v)$ must be an interior vertex of Γ'. Additionally, f is n-to-1, for some fixed n, on the neighbors of v (thus also the edges incident to v). Combining this with the fact that f preserves conductances,

$$\begin{aligned}\sum_{e,w:v\sim e\sim w} \gamma(e)(u(f(v)) - u(f(w))) &= \sum_{e',w':f(v)\sim e'\sim w'} n\gamma(e')(u(f(v)) - u(w')) \\ &= n \sum_{e',w':f(v)\sim e'\sim w'} \gamma(e')(u(f(v)) - u(w')) \\ &= 0.\end{aligned}$$

Having shown this, it follows straightforwardly that $\mathcal{U}_{-,M}$ is a contravariant functor, since for network morphisms $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma''$, $(g \circ f)^*(u) = u \circ g \circ f = g^* u \circ f = f^*(g^*(u))$. □

In this paper, we are primarily interested in how $\mathcal{U}$ acts as a functor on R-modules, with Γ fixed.

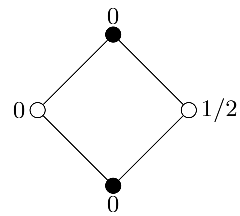
Proposition 1.3. *$\mathcal{U}_{\Gamma,-}$ is left exact.*

Proof. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Then since the submatrix $\begin{pmatrix} B^T & C \end{pmatrix}$ of the Kirchhoff matrix is a linear map, this diagram commutes (where the horizontal maps are simply coordinatewise applications of those in the original sequence):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^V & \longrightarrow & M^V & \longrightarrow & N^V & \longrightarrow & 0 \\ & & \big\downarrow (B^T\,C)_L & & \big\downarrow (B^T\,C)_M & & \big\downarrow (B^T\,C)_N & & \\ 0 & \longrightarrow & L^{\operatorname{int} V} & \longrightarrow & M^{\operatorname{int} V} & \longrightarrow & N^{\operatorname{int} V} & \longrightarrow & 0 \end{array}$$

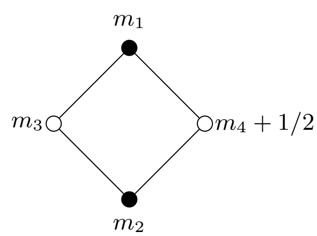
The rows are still exact and the kernels of the vertical maps are $\mathcal{U}_{\Gamma,L}$, $\mathcal{U}_{\Gamma,M}$, and $\mathcal{U}_{\Gamma,N}$, hence by the Snake Lemma, $0 \to \mathcal{U}_{\Gamma,L} \to \mathcal{U}_{\Gamma,M} \to \mathcal{U}_{\Gamma,N}$ is exact. □

However, $\mathcal{U}$ is not right exact, because it may not preserve the surjectivity of a map. Consider the $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ with $\pi : \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ the natural projection homomorphism and let Γ be the following network (with all conductances 1):



Displayed on the graph is a potential function taking values in $\mathbb{Q}/\mathbb{Z}$. This function is harmonic, since the net current at the left interior vertex is $0+0=0$ and the net current at the right interior vertex is $1/2+1/2=0$.

However, any $\mathbb{Q}$-valued harmonic function that projected to this one would have to be of the form



for integers m_1, m_2, m_3, m_4, and for it to be harmonic would require (in $\mathbb{Q}$)

$$m_3 - m_1 + m_3 - m_2 = 0$$
$$m_4 + 1/2 - m_1 + m_4 + 1/2 - m_2 = 0$$

or, rearranging into the kind of form that we will use for the rest of the paper,

$$2m_3 = m_1 + m_2$$
$$2m_4 + 1 = m_1 + m_2$$

This would imply that an even and odd integer are equal, which is a contradiction. So this particular $\mathbb{Q}/\mathbb{Z}$-valued harmonic function is not the image of a $\mathbb{Q}$-valued one under the map induced by $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. We say that this function **does not lift**. In the remainder of this paper, we will explore the implications of this.

2 The Harmonic Cohomology Module

2.1 Derived Functors

Given a left exact functor between categories of modules, we can construct derived functors through a process described in the Appendix, which describe the failure of the functor to be exact. In this section, we will take a first look at the derived functors of the functor $\mathcal{U}_{\Gamma,-}$.

In most of this paper, we will be considering for our ring a principal ideal domain R and using R for our module (over itself). In this case, Proposition A.2 gives that we have a simple injective resolution and thus an easy description of the derived functors.

Proposition 2.1. *For R a PID, F its field of fractions, and $\pi : F \to F/R$ the projection onto the quotient module, consider the induced map $\pi_* : \mathcal{U}_{\Gamma,F} \to \mathcal{U}_{\Gamma,F/R}$. Then the first derived functor of $\mathcal{U}_{\Gamma,-}$ at R is given by*

$$\mathcal{U}^1_{\Gamma,R} \cong \frac{\mathcal{U}_{\Gamma,F/R}}{\operatorname{im}\pi_*}$$

Additionally, $\mathcal{U}^j_{\Gamma,R} \cong 0$ for $j \geq 2$.

Proof. By Proposition A.2, an injective resolution of R is given by $0 \to R \to F \to F/R \to 0 \to 0 \to \dots$ To obtain the derived functors, we apply the functor to the terms of the sequence, omitting 0, to get the cochain complex $0 \to \mathcal{U}_{\Gamma,F} \xrightarrow{\pi_*} \mathcal{U}_{\Gamma,F/R} \to 0 \to 0 \to \dots$

The first derived functor is then defined to be the quotient of the kernel of the third arrow (all of $\mathcal{U}_{\Gamma,F/R}$) by the image of the second arrow (that is, $\operatorname{im}(\pi_*)$). This gives $\mathcal{U}^1_{\Gamma,R}$. The higher derived functors are then given by the cohomology at terms which are 0, which is trivially 0.

□

With this statement, we can now see that the harmonic function shown at the end of the previous section, a $\mathbb{Q}/\mathbb{Z}$-valued function on the network Γ which is not in the image of the map π_*, represents a non-identity element of $\mathcal{U}^1_{\Gamma,\mathbb{Z}}$.

For a fixed network Γ with conductances in the ring R, the module $\mathcal{U}^1_{\Gamma,R}$ is of most interest to us, and we refer to it as the **discrete harmonic cohomology module**, or just the **harmonic cohomology**. (Similarly, $U^j_{\Gamma,R}$ can be referred to as the **jth harmonic cohomology**.) In the next section, we will show an example of computing harmonic cohomology from the definition.

2.2 $\mathcal{U}^1_{K_{m,n},\mathbb{Z}}$

As a practice calculation we will compute the first cohomology module for the **complete bipartite bgraph** $K_{m,n}$. We say a bgraph is **bipartite** if there are no interior to interior or boundary to boundary edges. Thus the complete bipartite bgraph $K_{m,n}$ is the bipartite bgraph with m boundary vertices and n interior vertices, and every possible interior to boundary edge. To make the calculation easier, we will prove the following Lemma first. The basic idea is that since C is invertible, given any $\mathbb{Q}/\mathbb{Z}$ harmonic function we can find a $\mathbb{Q}$ harmonic function with the same boundary values; so up to the image of a $\mathbb{Q}$ harmonic function, each $\mathbb{Q}/\mathbb{Z}$ harmonic function is 0 on the boundary.

We will use the slightly informal notation $\mathcal{U}_{\Gamma,M}^{\text{condition}}$ to denote the collection of all M harmonic functions on Γ that satisfy "condition."

Lemma 2.2. *Let R be a PID, F its field of fractions, and $\pi : F^V \to (F/R)^V$ the natural R-module homomorphism. Suppose Γ is a Dirichlet non-singular network with unit conductances in R. Then*

$$\mathcal{U}_{\Gamma,R}^1 \cong \frac{\mathcal{U}_{\Gamma,F/R}^{0\ on\ bdry}}{\pi\left(\mathcal{U}_{\Gamma,F}^{R\ on\ bdry}\right)}$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{U}_{\Gamma,F}^{R \text{ on bdry}} & \longrightarrow & \mathcal{U}_{\Gamma,F} & \longrightarrow & (F/R)^{\partial V} & \longrightarrow & 0 \\ & & \downarrow \pi & & \downarrow \pi & & \downarrow \text{id} & & \\ 0 & \longrightarrow & \mathcal{U}_{\Gamma,F/R}^{0 \text{ on bdry}} & \longrightarrow & \mathcal{U}_{\Gamma,F/R} & \longrightarrow & (F/R)^{\partial V} & \longrightarrow & 0 \end{array}$$

The maps are the natural ones. We claim the rows are exact. Exactness is easy to see everywhere except at the rightmost points of both rows. To check exactness there, we first note the map $\mathcal{U}_{\Gamma,F} \to F^{\partial V}$ is surjective since the Dirichlet problem has a solution, and the projection map $F^{\partial V} \to (F/R)^{\partial V}$ is surjective, hence the composition $\mathcal{U}_{\Gamma,F} \to (F/R)^{\partial V}$ is surjective. Hence exactness of the top row. Now this map $\mathcal{U}_{\Gamma,F} \to (F/R)^{\partial V}$ is the composition of two maps $\mathcal{U}_{\Gamma,F} \to U_{\Gamma,F/R} \to (F/R)^{\partial V}$, and the whole thing being surjective implies the second map is surjective, which gives surjectivity of the bottom row.

Thus we may apply the Snake Lemma, and since the furthest right vertical map has trival kernel and cokernel, we get that

$$0 \to \frac{\mathcal{U}_{\Gamma,F/R}^{0 \text{ on bdry}}}{\pi\left(\mathcal{U}_{\Gamma,F}^{R \text{ on bdry}}\right)} \to \mathcal{U}_{\Gamma,R}^1 \to 0$$

is exact. This gives the result. □

Theorem 2.3. *Denote by $K_{m,n}$ the network with m boundary vertices, n interior vertices, and every possible interior to boundary edge, all with conductances $1 \in \mathbb{Z}$. Then we have that*

$$\mathcal{U}_{K_{m,n},\mathbb{Z}}^1 \cong (\mathbb{Z}/m)^{n-1}.$$

Proof. Since the bgraph is bipartite, C is invertible over $\mathbb{Q}$, and thus by Lemma 2.2 we can assume all $\mathbb{Q}/\mathbb{Z}$ harmonic functions are 0 on the boundary. So for each such function, it must be that for all $v \in \operatorname{int} V$

$$mu(v) = \sum_{w \sim v} u(w) = 0 \quad \text{ in } \mathbb{Q}/\mathbb{Z}.$$

The above equation has solutions $u(v) = 0, 1/m, \ldots, (m-1)/m$ in $\mathbb{Q}/\mathbb{Z}$. Since this equation holds for all $v \in \operatorname{int} V$ and each is independent of the others, it follows that

$$\mathcal{U}^{0 \text{ on bdry}}_{\Gamma, F/R} \cong (Z/m)^n.$$

Now suppose u is a $\mathbb{Q}$ harmonic function that has values in R on the boundary, then we again get that it is necessary and sufficient for it to satisfy, for each $v \in \operatorname{int} V$,

$$mu(v) = \sum_{w \sim v} u(w) = \sum_{w \in \partial V} u(w) \quad \text{ in } \mathbb{Q}.$$

But since this is in $\mathbb{Q}$, it follows that u is constant on the interior. Applying the same reasoning as before when we project from F to F/R, we then get that $u(v) = 0, 1/m, \ldots, (m-1)/m$ for *all* $v \in \operatorname{int} V$. Thus

$$\pi\left(\mathcal{U}^{R \text{ on bdry}}_{\Gamma, F}\right) \cong \mathbb{Z}/m.$$

Thus by Lemma 2.2 we get the result. ☐

2.3 The Cokernel Interpretation

As we saw in the previous section, computing harmonic cohomology from the definition is somewhat unwieldy, and that particular demonstration was dependent on the specific family of graphs. Using the snake lemma, we can describe the cohomology in terms of the cokernel of a matrix, allowing us to consider it using the tools of linear algebra and modules.

Theorem 2.4. *Suppose R is a PID, and F is its field of fractions. Consider some network Γ such that $\begin{pmatrix} B^T & C \end{pmatrix}$ is surjective as a map $F^V \to F^{\operatorname{int} V}$ (in particular, this holds for any Dirichlet nonsingular network.) Then*

$$\mathcal{U}^1_{\Gamma, R} \cong \operatorname{coker} \begin{pmatrix} B^T & C \end{pmatrix}_R$$

where the matrix $(B^T\, C)_R$ is interpreted as a map $R^V \to R^I$.

Proof. We have the following commutative diagram, where f, g, h are the natural projections:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{U}_{\Gamma,F} & \longrightarrow & F^V & \xrightarrow{(B^T\,C)} & F^{intV} & \longrightarrow & 0 \\
 & & \big\downarrow f & & \big\downarrow g & & \big\downarrow h & & \\
0 & \longrightarrow & \mathcal{U}_{\Gamma,F/R} & \longrightarrow & (F/R)^V & \xrightarrow[(B^T\,C)]{} & (F/R)^{intV} & \longrightarrow & 0
\end{array}$$

Certainly the inclusion maps $\mathcal{U}_{\Gamma,F} \to F^V$ and $\mathcal{U}_{\Gamma,F/R} \to (F/R)^V$ are injective. The exactness of the upper row at F^V follows from the definition of $\mathcal{U}_{\Gamma,F}$ as a kernel, and similarly for the lower row at $(F/R)^V$. To finish showing exactness of the rows, it then suffices to show that the two maps given by $\begin{pmatrix} B^T & C \end{pmatrix}$ are surjective. Since $\det(C)$ is nonzero, it is a unit in F and hence C is invertible as a matrix over F. Then for any $u \in F^{\mathrm{int}\,V}$, there is a vector $C^{-1}u$ such that $\begin{pmatrix} B^T & C \end{pmatrix} \begin{pmatrix} 0 & C^{-1}u \end{pmatrix}^T = u$, so the upper map is surjective. Now, for any $u \in (F/R)^{\mathrm{int}\,V}$, since h is surjective we can pull u back to an element in $F^{\mathrm{int}\,V}$, which we can pull back to an element in F^V and then map to some element $v \in (F/R)^V$ by g. By commutativity, v is mapped by $(B^T\ C)$ to u. Hence, the lower map is also surjective. Thus, the Snake Lemma yields the following exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0$$

But now observe that both h and g are onto and thus $\operatorname{coker} g \cong \operatorname{coker} h \cong 0$. It is also easy to see that $\ker g \cong R^V$ and $\ker h \cong R^{intV}$. Also, $\operatorname{coker} f = \mathcal{U}^1_{\Gamma,R}$. Thus we get the following arrows

$$R^V \xrightarrow{(B^T\,C)_R} R^{intV} \twoheadrightarrow \mathcal{U}^1_{\Gamma,R}$$

So from the first isomorphism theorem and exactness we get that

$$\mathcal{U}^1_{\Gamma,R} \cong R^{intV} / \ker(R^{intV} \to \mathcal{U}^1_{\Gamma,R}) \cong R^{intV} / \operatorname{im}(B^T\,C)_R.$$

which is the required cokernel. $\square$

While this formulation does not apply if $\begin{pmatrix} B^T & C \end{pmatrix}$ is not surjective over F, it still covers a wide swath of cases. As a concluding example, we will repeat the calculation of $\mathcal{U}^1_{K_{m,n},\mathbb{Z}}$ using its matrix.

First, for $\Gamma = K_{m,n}$ with conductances 1, the matrix is given by

$$\left(\begin{array}{cccc|cccc} -1 & -1 & \dots & -1 & m & 0 & \dots & 0 \\ -1 & -1 & \dots & -1 & 0 & m & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \dots & -1 & 0 & 0 & \dots & m \end{array}\right)$$

As long as we remain in $\mathbb{Z}$, performing column reduction operations on the matrix does not change its column space, and thus does not change the cokernel. We can thus subtract the first column from all of the others in the left block to make them 0:

$$\left(\begin{array}{cccc|cccc} -1 & 0 & \dots & 0 & m & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 0 & m & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & \dots & 0 & 0 & 0 & \dots & m \end{array}\right)$$

In determining the column space, we can ignore 0 columns, so we can freely delete these from the matrix. Then we can add m times the leftmost column to the first column of the right block to get

$$\left(\begin{array}{c|cccc} -1 & 0 & 0 & \dots & 0 \\ -1 & -m & m & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -m & 0 & \dots & m \end{array}\right)$$

By adding the third, fourth,... columns to the second, we cancel out all of its nonzero coefficients, reducing it to 0. Removing this column gives

$$\left(\begin{array}{c|ccc} -1 & 0 & \dots & 0 \\ -1 & m & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & \dots & m \end{array}\right)$$

Since we obtained this matrix from the $\begin{pmatrix} B^T & C \end{pmatrix}$ matrix of $K_{m,n}$ through column operations, they have the same column space and cokernel. The column space of this matrix is all vectors whose entries are all equal mod m. An equivalence class modulo this image is then defined by the differences between adjacent entries mod m. There are $n-1$ of these differences for an n-dimensional vector, so the cokernel is $(\mathbb{Z}/m\mathbb{Z})^{n-1}$.

3 Transformations of Boundary Graphs

In analogy with other cohomology theories, we expect the discrete harmonic cohomology to reflect the "topology" of a network. In this section, we show a basic motivating result in this area.

Conjecture 3.1. *A bgraph is quasi-layerable if and only if for any ring R, any R-module M, any network Γ given by the bgraph with unit conductances in R, and any $j \geq 1$, $\mathcal{U}^j_{\Gamma,M} \cong 0$.*

3.1 Layerability

Lemma 3.2. *If Γ' is obtained from Γ with unit conductances by deleting a boundary edge or boundary spike, then the inclusion $\Gamma' \to \Gamma$ induces a natural isomorphism $\mathcal{U}_{\Gamma,M} \to \mathcal{U}_{\Gamma',M}$.*

Proof. When Γ' is obtained from Γ by removing a boundary edge, every harmonic potential u' on Γ' extends to a unique harmonic potential u on Γ, namely the one that takes the exact same values as u'. Now consider when Γ' is obtained from Γ by removing a boundary spike. Denote the boundary and interior vertices associated with the spike as b and i, respectively. Suppose $u' \in \mathcal{U}_{\Gamma',M}$. We want to show that u' extends to a unique harmonic potential $u \in \mathcal{U}_{\Gamma,M}$.

Notice that the potential is determined at all vertices of Γ except for b. In order to satisfy Kirchhoff's Current Law, we must define $u(b)$ such that

$$\sum_{v\sim i}\gamma(iv)(u(i)-u(v))=\gamma(ib)(u(i)-u(b))+\sum_{v\sim i,v\neq b}\gamma(iv)(u(i)-u(v))=0.$$

Hence, since $\gamma(ib)$ is a unit, we must set

$$u(b)=u(i)+\frac{1}{\gamma(ib)}\sum_{v\sim i,v\neq b}\gamma(iv)(u(i)-u(v)).$$

The reader may verify that in either case, the isomorphism $\mathcal{U}_{\Gamma,M}\to\mathcal{U}_{\Gamma',M}$ is natural. $\square$

Notice that this is the only time we actually use the fact that conductances are units. However, the implications of this lemma justify our restriction of conductances to units.

Lemma 3.3. *If Γ' is obtained from Γ by deleting a boundary edge, boundary spike, or isolated boundary vertex, then the inclusion $\Gamma'\to\Gamma$ induces an isomorphism $\mathcal{U}^j_{\Gamma,M}\to\mathcal{U}^j_{\Gamma',M}$ for all $j\geq 1$.*

Proof. Take an injective resolution of M:

$$0\to M\to C^0\to C^1\to\dots$$

We will first consider the cases of deleting a boundary spike or boundary edge. Since $\Gamma'\to\Gamma$ induces a natural isomorphism $\mathcal{U}_{\Gamma,N}\to\mathcal{U}_{\Gamma',N}$ for any R-module N, we have an isomorphism of cochain complexes between $\mathcal{U}_{\Gamma,C^\bullet}$ and $\mathcal{U}_{\Gamma',C^\bullet}$. Hence, since functors preserve isomorphisms, for any $j\geq 1$ we have

$$\mathcal{U}^j_{\Gamma,M}\cong H^j(\mathcal{U}_{\Gamma,C^\bullet})\cong H^j(\mathcal{U}_{\Gamma',C^\bullet})\cong\mathcal{U}^j_{\Gamma',M}.$$

Now we consider the case of removing an isolated boundary vertex. The inclusion $\Gamma'\to\Gamma$ induces a surjection $\mathcal{U}_{\Gamma,C^j}\twoheadrightarrow\mathcal{U}_{\Gamma',C^j}$ since we can extend any harmonic function on Γ' to a harmonic function on Γ by setting the potential at the isolated boundary vertex to be any value. Notice that if $\mathcal{U}_{\Gamma,C^j}\to\mathcal{U}_{\Gamma',C^j}$ is defined by projecting a harmonic function on Γ onto Γ', $\ker(\mathcal{U}_{\Gamma,C^j}\to\mathcal{U}_{\Gamma',C^j})$ is isomorphic to the module of all possible values that the isolated boundary vertex can take, which is C^j. We can then construct a short exact sequence of cochain complexes:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^0 & \longrightarrow & \mathcal{U}_{\Gamma,C^0} & \longrightarrow & \mathcal{U}_{\Gamma',C^0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^1 & \longrightarrow & \mathcal{U}_{\Gamma,C^1} & \longrightarrow & \mathcal{U}_{\Gamma',C^1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

This gives us a long exact sequence on cohomology

$$\begin{aligned}0 &\to H^0(C^\bullet) \to \mathcal{U}_{\Gamma,M} \to \mathcal{U}_{\Gamma',M} \to 0 \to \mathcal{U}^1_{\Gamma,M} \to \mathcal{U}^1_{\Gamma',M} \to 0 \\ &\to \dots \to 0 \to \mathcal{U}^j_{\Gamma,M} \to \mathcal{U}^j_{\Gamma',M} \to 0 \to \dots\end{aligned}$$

since $H^j(C^\bullet) \cong 0$ for $j \geq 1$. Thus, $\mathcal{U}^j_{\Gamma,M} \cong \mathcal{U}^j_{\Gamma',M}$ for $j \geq 1$. □

3.2 Pasting Networks

Given some collection of bgraphs, an **m-pasting** of them is obtained by choosing an m-tuple of distinct boundary vertices from each one, then identifying the vertices in each position of all the tuples.

Lemma 3.4. *Suppose M is an R-module, Γ' and Γ'' are networks and Γ is a 1-pasting of Γ' and Γ''. Then for all $j \geq 1$,*

$$\mathcal{U}^j_{\Gamma,M} = \mathcal{U}^j_{\Gamma',M} \oplus \mathcal{U}^j_{\Gamma'',M}$$

Proof. Let $\iota : \mathcal{U}^1_{\Gamma,M} \hookrightarrow \mathcal{U}^1_{\Gamma',M} \oplus \mathcal{U}^1_{\Gamma'',M}$ be the map obtained by defining the functions on Γ' and on Γ'' using the relevant values on the larger graph. (In particular, the two functions will agree on the pasted boundary vertex.) Since every vertex of Γ is represented in Γ' or Γ'' with its value unchanged, this map is injective.

Take an injective resolution of M:

$$0 \to M \to C^0 \to C^1 \to \dots$$

Notice that for any module N, $\operatorname{coker}(\iota)$ is the quotient of $\mathcal{U}_{\Gamma',N} \oplus \mathcal{U}_{\Gamma'',N}$ by the the pairs of harmonic functions which have the same voltage on the shared boundary vertex. This is just all possible values by which the two harmonic functions can differ at the shared boundary vertex. Since adding a constant value to the potential at each vertex results in another harmonic function (since the definition of harmonicity depends only on the differences between potentials), we can get any value in N. So, $\operatorname{coker}(\iota) \cong N$. Then since Γ' and Γ'' are sub-bgraphs of Γ, we have the short exact sequence of cochain complexes:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{U}_{\Gamma,C^0} & \longrightarrow & \mathcal{U}_{\Gamma',C^0} \oplus \mathcal{U}_{\Gamma'',C^0} & \longrightarrow & C^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{U}_{\Gamma,C^1} & \longrightarrow & \mathcal{U}_{\Gamma',C^1} \oplus \mathcal{U}_{\Gamma'',C^1} & \longrightarrow & C^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

This induces a long exact sequence on cohomology

$$0 \to H^0(\mathcal{U}_{\Gamma,C^\bullet}) \to H^0(\mathcal{U}_{\Gamma',C^\bullet} \oplus \mathcal{U}_{\Gamma'',C^\bullet}) \to H^0(C^\bullet) \to H^1(\mathcal{U}_{\Gamma,C^\bullet})$$
$$\to \cdots \to H^{j-1}(C^\bullet) \to H^j(\mathcal{U}_{\Gamma,C^\bullet}) \to H^j(\mathcal{U}_{\Gamma',C^\bullet} \oplus \mathcal{U}_{\Gamma'',C^\bullet}) \to H^j(C^\bullet) \to \dots$$

Since cohomology commutes with direct sum,

$$H^j(\mathcal{U}_{\Gamma',C^\bullet} \oplus \mathcal{U}_{\Gamma'',C^\bullet}) \cong H^j(\mathcal{U}_{\Gamma',C^\bullet}) \oplus H^j(\mathcal{U}_{\Gamma'',C^\bullet}) \cong \mathcal{U}^j_{\Gamma',M} \oplus \mathcal{U}^j_{\Gamma'',M},$$

and for any $j \geq 2$, $H^{j-1}(C^\bullet) \cong 0 \cong H^j(C^\bullet)$. Hence, for each $j \geq 1$ we have the exact sequence

$$0 \to \mathcal{U}^j_{\Gamma,M} \to \mathcal{U}^j_{\Gamma',M} \oplus \mathcal{U}^j_{\Gamma'',M} \to 0$$

Thus, $H^j(\mathcal{U}_{\Gamma,C^\bullet}) \cong H^j(\mathcal{U}_{\Gamma',C^\bullet}) \oplus H^j(\mathcal{U}_{\Gamma'',C^\bullet})$.

□

Corollary 3.5. *If the graph Γ' is obtained from Γ by deleting an interior spike, then $\mathcal{U}^j_{\Gamma',M} \cong \mathcal{U}^j_{\Gamma,M}$ for all $j \geq 1$.*

Proof. Recall that an interior spike is obtained simply by 1-pasting the graph with one boundary vertex and one adjacent interior vertex, so the cohomology of the graph with the spike added is obtained from the original one by taking the direct sum with a trivial module. □

Now suppose we have a finite abelian group $\mathbb{Z}/({p_1}^{\alpha_1}) \times \mathbb{Z}/({p_2}^{\alpha_2}) \times \ldots \times \mathbb{Z}/({p_k}^{\alpha_k})$. We already know that there is a network Γ^i such that $\mathcal{U}^1_{\Gamma^i,\mathbb{Z}} \cong \mathbb{Z}/({p_i}^{\alpha_i})$, namely $K_{p_i^{\alpha_i},2}$ with conductances all 1 in $\mathbb{Z}$. We can then take a 1-pasting of $\Gamma^1, \ldots, \Gamma^k$ to form a new network Γ. By Lemma 3.4, $\mathcal{U}^1_{\Gamma,\mathbb{Z}} \cong \mathbb{Z}/({p_1}^{\alpha_1}) \times \mathbb{Z}/({p_2}^{\alpha_2}) \times \ldots \times \mathbb{Z}/({p_k}^{\alpha_k})$. This proves the following theorem.

Theorem 3.6. *Given any finite abelian group M, there is a network Γ such that $\mathcal{U}^1_{\Gamma,\mathbb{Z}} \cong M$.*

Consolidating the above results, we can get an algebraic consequence of quasi-layerability.

Lemma 3.7. *A graph has a boundary cutpoint if and only if it is a nontrivial 1-pasting of graphs, in which case it is specifically a pasting of the graphs formed by splitting the cutpoint along the vertex corresponding to it.*

Proof. First, assume a graph has a boundary cutpoint. Then in each of the component graphs formed by splitting it, select the boundary vertex corresponding to the boundary cutpoint in the original graph, and identify them together. Then the resulting graph has exactly the same vertices as the original graph; all edge relationships within the components remain the same, and the pasted boundary vertex has all of the edge relationships of the original boundary cutpoint restored.

Similarly, assume a graph is formed by 1-pasting other graphs; then if the boundary vertex at which the pasting occurred is removed, vertices originating in different pasted graphs will no longer be connected, so the vertex is a cutpoint.

□

Theorem 3.8. *Let R be a ring, M an R-module, and Γ a network with unit conductances in R. If the underlying graph of Γ is quasi-layerable (in particular, if it is layerable), then $\mathcal{U}^j_{\Gamma,M} \cong 0$ for any $j \geq 1$.*

Proof. By induction on $|V| + |E|$. The base case is the graph with a single isolated boundary vertex, on which harmonic functions are simply elements of M, so its cohomology is trivial. Then assuming the proposition for $|V| + |E| \leq n$, consider a quasi-layerable graph with $|V| + |E| = n + 1$. Then we can either contract a spike, delete an edge, delete an isolated vertex, or split a boundary cutpoint (thus expressing the cohomology as a direct sum of those of the components) to produce a quasi-layerable graph with $|V| + |E|$ strictly smaller. By the induction hypothesis, the resulting graph or graphs have trivial cohomology, so by the induction hypothesis, Lemma 3.3, and Lemma 3.4, $\mathcal{U}^j_{\Gamma,M}$ is a direct sum of trivial modules and is trivial. □

This result is somewhat remarkable, because for a quasi-layerable boundary graph, *any* assignment of unit conductances will produce a network with trivial harmonic cohomology.

We would like it if the geometric property of quasi-layerability were actually equivalent to this criterion of all cohomology modules for all networks on the graph being trivial. Having proven one direction of Conjecture 3.1, we now state the converse as a conjecture:

Conjecture 3.9. *Given a bgraph G, if $\mathcal{U}^j_{\Gamma,M} \cong 0$ for every network Γ on G, ring R, R-module M, and $j \geq 1$, then G is quasi-layerable.*

However, the hypothesis of this statement seems too general to work with. Instead, we focus our attention on a much stronger statement:

WILL: From this point on, we need to edit this to shift the focus onto local rings.

Conjecture 3.10. *There exists a ring R (the "one ring to rule them all") such that if $\mathcal{U}^1_{\Gamma,R} \cong 0$ for all networks Γ on a bgraph G, then G is quasi-layerable.*

Our current candidate for the ring R is the ring of Eisenstein integers: $\mathbb{Z}[\zeta_3]$, where $\zeta_3 = e^{2\pi i/3}$. The reason for choosing this ring, and a description of our progress on this conjecture, are explained in the next section.

Lemma 3.11. *Suppose v is an interior vertex in a network Γ such that it has no neighbors in $\operatorname{int} V$, the sum of the conductances of incident edges is 0, and changing v to a boundary vertex makes Γ layerable. Then*

$$\mathcal{U}^1_{\Gamma,R} \cong \frac{\mathcal{U}^{0 \text{ everywhere else}}_{\Gamma,F/R}}{\pi(\mathcal{U}^{R \text{ everywhere else}}_{\Gamma,F})},$$

where π is the projection $\mathcal{U}_{\Gamma,F} \to \mathcal{U}_{\Gamma,F/R}$.

Proof. □

STAR PASTING

4 Making $\mathcal{U}^1_{\Gamma,R}$ Nontrivial

Conjecture 3.10 can be restated in the contrapositive:

Conjecture 4.1. *Given any non-quasi-layerable bgraph, there is an assignment of unit conductances in some ring R (specifically, $\mathbb{Z}[\zeta_3]$) defining a network Γ with $\mathcal{U}^1_{\Gamma,R}$ nontrivial.*

In this section, we refer to such an assignment as a **nontrivial labeling in R**.

This suggests attempting to prove the conjecture by finding an algorithmic way of assigning these conductances. While such an algorithm appears difficult to find in general, results on certain classes of bipartite graphs have been obtained. (Here and elsewhere, a bipartite bgraph refers to one with no edges between boundary vertices or between interior vertices; we never consider any other sort of bipartition.) Overall, the case of bipartite graphs appears to be significantly easier; this is related to the C block of the Kirchhoff matrix being diagonal, since its off-diagonal entries describe interior-interior edges.

4.1 Why $\mathbb{Z}[\zeta_3]$?

We choose to work in the ring $\mathbb{Z}[\zeta_3]$ because it is a PID with a useful property that $\mathbb{Z}$ lacks:

Proposition 4.2. *Each unit in $\mathbb{Z}[\zeta_3]$ can be expressed as a sum of two units.*

Proof. The units in $\mathbb{Z}[\zeta_3]$ are 1, ζ_3, ζ_3^2, -1, $-\zeta_3$, and $-\zeta_3^2$. Additionally, since $1+\zeta_3+\zeta_3^2=0$, $1=-\zeta_3-\zeta_3^2$. Having expressed 1 in this way, we can multiply both sides by any other unit, which gives the result. $\square$

When constructing nontrivial labellings, we often add up all the unit conductances incident to a vertex; this property allows us to switch the parity of the resulting sum. The value of this will become clear in the following proofs.

Having said this, we have not yet explicitly found a bgraph which does not have a nontrivial labeling in $\mathbb{Z}$, and whether one exists is an interesting question; it's possible that the ring $\mathbb{Z}$ actually suffices to determine quasi-layerability. However, $\mathbb{Z}[\zeta_3]$ seems easier to work with systematically.

4.2 Maximal Minors

As we have seen, it is usually easier to analyze the harmonic cohomology using the cokernel of $\begin{pmatrix} B^T & C \end{pmatrix}$. This opens it up to the tools of linear algebra and modules, in particular one lemma presented below.

Lemma 4.3. *Given an $m \times n$ matrix A with entries in some commutative ring R defining a module homomorphism $f : R^n \to R^m$, it is surjective (so its cokernel is trivial) if and only if its $m \times m$ minors generate the entire ring.*

Proof. (due to Math StackExchange user Martin Brandenburg [1])

Note that surjectivity is equivalent to the existence of a homomorphism g such that $fg = \mathrm{id}$, represented by an $n \times m$ matrix B over R such that $AB = I_m$.

First assume surjectivity. Then, the right inverse B exists. For each $m \times m$ submatrix Y_i of A formed by taking some subset of the columns, let Z_i be the corresponding submatrix of B formed by taking the rows with the same indices. The Cauchy-Binet formula then gives that

$$1 = \det(AB) = \sum_{i=1}^{k} \det(Y_i)\det(Z_i)$$

Then since there is a combination of the minors $\det(Y_i)$ with coefficients in R (the $\det(Z_i)$) equaling 1, the entire ring can be generated by the $m \times m$ minors.

Now instead assume that the minors generate the whole ring; then there are coefficients λ_i such that $\sum_{i=1}^{k} \lambda_i \det(Y_i) = 1$. For each Y_i, construct an $n \times m$ matrix B_i in the following way: at each row index of B_i corresponding to one of the columns used in Y_i, insert the corresponding column of $\mathrm{adj}(Y_i)$, the adjugate of Y_i, and make the remaining rows all 0. Then consider the matrix $B = \sum_{i=1}^{k} \lambda_i B_i$; we will compute AB. First notice that, because of the zero columns in B_i, $AB_i = Y_i B_i$. Additionally, it is a property of the adjugate that $Y_i \,\mathrm{adj}(Y_i) = \det(Y_i) I_m$. So

$$AB = \sum_{i=1}^{k} \lambda_i AB_i = \sum_{i=1}^{k} \lambda_i Y_i B_i = \sum_{i=1}^{k} \lambda_i \det(Y_i) I_m = I_m \qquad \square$$

Since the matrix $\begin{pmatrix} B^T & C \end{pmatrix}$ is always wider than it is tall, the $m \times m$ minors are the maximal minors, and we will refer to them as such.

This lemma then reduces the conjecture, in many cases, to finding a way to fill in the entries of $\begin{pmatrix} B^T & C \end{pmatrix}$ such that all the maximal minors have a common factor. With this approach in mind, we can already demonstrate nontrivial labelings for a class of graphs.

Theorem 4.4. *Given a bipartite bgraph G with no interior spikes and $|\partial V| \leq |\operatorname{int} V|$, G has a nontrivial labeling in $\mathbb{Z}[\zeta_3]$.*

Proof. Since the graph is bipartite, the edges incident to each interior vertex are distinct from the edges incident to any other interior vertex. As such, we can assign conductances in the following way:

- At each interior vertex of even degree, give all edges conductance 1.
- Each interior vertex of odd degree has degree at least 3, since there are no spikes. Assign conductance $-\zeta_3$ to one edge and $-\zeta_3^2$ to another, and 1 to everything else.

Since $-\zeta_3 - \zeta_3^2 = 1$, the sum of this with an odd number of 1s is even. So under this assignment, the sum of the conductances incident to each interior vertex is even and nonzero. Thus C will be a diagonal matrix with even, nonzero entries. This implies the new network Γ is Dirichlet nonsingular.

Since $\begin{pmatrix} B^T & C \end{pmatrix}$ is an $|\operatorname{int} V| \times |V|$ matrix, its maximal minors are obtained by picking $|\operatorname{int} V|$ columns and taking the determinant of the resulting submatrix. Since B^T has $|\partial V|$ columns, if $|\partial V| < |\operatorname{int} V|$, every maximal minor must contain a column from C. Since every column of C contains only a single, even entry, cofactor expansion gives that every maximal minor must be divisible by 2.

If $|\partial V| = |\operatorname{int} V|$, there is a single maximal minor which does not contain any column from C, the determinant of B^T. Then the sum across each row is still even, since it is the sum of all conductances incident to a given interior vertex.

The determinant commutes with any ring homomorphism, so in particular the determinant of B^T mod 2 is equal to the determinant of B^T interpreted as a matrix over $\mathbb{Z}[\zeta_3]/(2)$. In this latter case, all the rows have entries summing to 0 mod 2, so the rows do not span the entire space, and the determinant is 0 mod 2. In this case, too, every maximal minor is even. Either way, by Lemma 4.3, $\begin{pmatrix} B^T & C \end{pmatrix}$ has nontrivial cokernel, and $\mathcal{U}^1_{\Gamma,\mathbb{Z}[\zeta_3]} \ncong 0$. ◻

Corollary 4.5. *Given a bipartite bgraph with s interior spikes and $|\partial V| \leq |\operatorname{int} V| - s$, it has a nontrivial labeling in $\mathbb{Z}[\zeta_3]$.*

Proof. The boundary cutpoints attaching each of the interior spikes can be split to give a bipartite graph with $|\operatorname{int} V| - s$ interior vertices, along with several 1-stars. Lemma 3.4 and the above theorem give the result. ◻

4.3 Bipartite Graphs with 3 Interior Vertices

While the above theorem is a first step, the class of graphs it treats is ultimately quite small. A more promising general approach is to examine ways of reducing bgraphs to smaller ones in ways beyond the quasi-layerability operations, and understand what effect this has on cohomology. Following is a lemma describing how we can use one such operation, and then an application of that lemma to show the conjecture for bipartite graphs with 3 interior vertices.

Lemma 4.6. *Given any bgraph G, consider the bgraph G' obtained by adding a new boundary vertex v', whose neighbors are exactly those of some other boundary vertex v. Then if there is a Dirichlet nonsingular nontrivial labeling of G over some ring containing nontrivial third roots of unity, there is one of G'.*

Proof. Given the network Γ produced by a Dirichlet nonsingular nontrivial labeling of G, transform it into a network Γ' on G' as follows:

- Multiply the conductances on every edge incident to v by $-\zeta_3$.
- Give each edge incident to v' the conductance of the original edge connecting v to the same vertex, multiplied by $-\zeta_3^2$.

Now consider the effect of this transformation on $\begin{pmatrix} B^T & C \end{pmatrix}$. First, the sums across the rows of B^T, and thus C, will not be changed: if $-\lambda_{vi}$ is the term in such a sum representing the conductance along the edge from v to some interior vertex i, it will be replaced by $\zeta_3\lambda_{vi} + \zeta_3^2\lambda_{vi} = -\lambda_{vi}$.

Then we can split the maximal minors of the new matrix $\begin{pmatrix} B'^T & C' \end{pmatrix}$ into 3 types:

- A maximal square submatrix may contain neither of the columns corresponding to v, v'; then the columns composing it also occur in the original $\begin{pmatrix} B^T & C \end{pmatrix}$, and the determinant is the same.
- The submatrix may contain exactly one of the columns; in this case, its columns will be the same as those of some maximal square submatrix in $\begin{pmatrix} B^T & C \end{pmatrix}$, except one will be multiplied by ζ_3 or ζ_3^2. Thus the determinant will be the same as some maximal minor of the original, up to multiplication by a unit.
- The submatrix may contain both of the columns; since they are scalar multiples of each other, the determinant is 0.

As a result, the set of maximal minors of $\begin{pmatrix} B'^T & C' \end{pmatrix}$ is, up to multiplication by units and 0, the same as the set of maximal minors of $\begin{pmatrix} B^T & C \end{pmatrix}$. Since Γ is given by a nontrivial labeling, by Lemma 4.3, the latter set generates a proper ideal, so the former set does as well, and Γ' is a nontrivial labeling of G'. □

Using this lemma, we can often limit our investigation to graphs whose boundary vertices all have distinct neighbors; however, this is not always the case. Collapsing boundary vertices with the same neighbors in a non-quasi-layerable graph can create boundary cutpoints, and may even render the graph quasi-layerable. This particular issue inspired the definition of quasi-layerability in the first place.

Theorem 4.7. *Every non-quasi-layerable bipartite graph with 3 or fewer interior vertices has a nontrivial labeling in* $\mathbb{Z}[\zeta_3]$.

Proof. For one interior vertex, the statement is vacuously true, since any boundary vertex would have to be a spike, which could then be contracted, so the graph would be layerable.

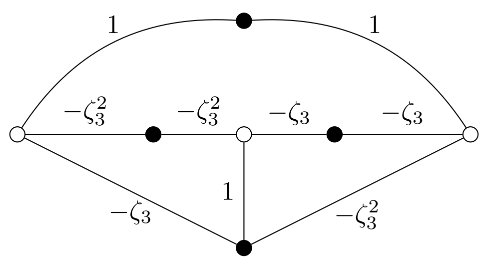
For two interior vertices, if any boundary vertex had degree 1, it could be contracted to produce a graph with one interior vertex. After deleting boundary edges, this would be layerable, by the above. So the only nonlayerable bipartite bgraphs with two interior vertices are those for which every boundary vertex has 2 neighbors; these are exactly the $K_{m,2}$, which we know from Theorem 2.3 have a nontrivial labeling (while the proof applied there was in $\mathbb{Z}$, it works identically over $\mathbb{Z}[\zeta_3]$).

Now consider a graph with 3 interior vertices and assume without loss of generality that there are no boundary spikes or interior spikes. Call the three interior vertices 1, 2, and 3; then associate to each boundary vertex the set of its neighbors. Since there are no boundary spikes, the possible sets of neighbors are {1,2}, {1,3}, {2,3}, and {1,2,3}.

Then we can classify graphs based on which of these sets occur among the boundary vertices. The proof proceeds by casework. In each case, however, we use the same induction procedure: first, demonstrate a nontrivial labeling for the non-quasi-layerable graph in the case with the fewest boundary vertices.

Then assume the statement is true for graphs of the class on n boundary vertices and consider a graph of the class with $n+1$ boundary vertices. If there are more boundary vertices than the base case, there must be two which have the same set of neighbors. This graph can be obtained from a graph of the class on n vertices by duplicating one of the boundary vertices, so by the induction hypothesis and Lemma 4.6, the result holds. As such, we need only approach the base cases separately.

- First, the case in which the graph has boundary vertices corresponding to all 4 sets. For the base case, consider this graph, with one boundary vertex representing each:



  Depicted on the graph is a nontrivial labeling, which can be seen by considering the matrix $\begin{pmatrix} B^T & C \end{pmatrix}$:

$$\begin{pmatrix} -1 & \zeta_3^2 & 0 & \zeta_3 & 2 & 0 & 0 \\ 0 & \zeta_3^2 & \zeta_3 & -1 & 0 & 2 & 0 \\ -1 & 0 & \zeta_3 & \zeta_3^2 & 0 & 0 & 2 \end{pmatrix}$$

  The sum of every column is divisible by 2, so any maximal submatrix is degenerate mod 2; thus every maximal minor is 0 mod 2. By Lemma 4.3, this is a nontrivial labeling.

- Suppose three sets are represented, either the three 2-element sets or two 2-element sets and the 3-element set. For the base cases, consider each of the graphs having one boundary vertex with each set of interior neighbors. These graphs have no interior spikes, since in all of the cases above, each interior vertex is contained in at least 2 sets; additionally, they have only 3 boundary vertices, so by Theorem 4.4 there is a nontrivial labeling.

- Suppose two two-element sets are represented. Unlike in the previous cases, some of the graphs of this type are quasi-layerable, specifically the

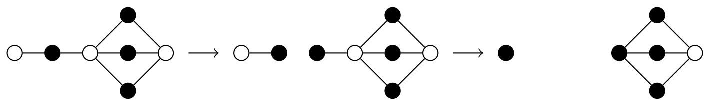
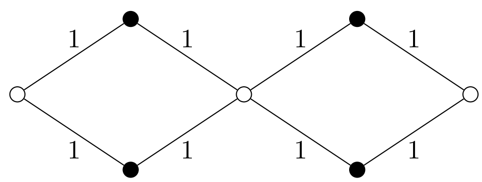


Figure 3: Reduction of the type of quasi-layerable graph discussed above.

ones for which there is only one boundary vertex corresponding to one of the sets (see Figure 4.3). Our induction must then run over graphs with at least two boundary vertices in each class. The base case is the graph with exactly two in each class:



Assigning conductance 1 everywhere produces for $\begin{pmatrix} B^T & C \end{pmatrix}$

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 4 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 2 \end{pmatrix}$$

As in the first case, the column sums are all even, so this is a nontrivial labeling. The induction proceeds as in the other cases, except that in the induction step, the identification of duplicate boundary vertices should never reduce the number of boundary vertices sharing those neighbors below 2.

- The penultimate case is that in which a two-element set and a three-element set are represented. The base case is the graph with one boundary vertex representing each, which is easily seen to be a $K_{2,2}$ with an interior spike; thus it has a nontrivial labeling.

- Finally, if only one set is represented, every boundary vertex has the same neighbors; thus the graph is a complete bipartite graph, so if it is non-quasi-layerable, it has a nontrivial labeling.

□

This method could, in theory, be extended to prove the result for general graphs on three interior vertices, or bipartite graphs with larger fixed numbers of interior vertices. However, while the former might be helpful, the latter is impractical because the number of graphs which need to be checked in the base case grows as $2^{2^{|\operatorname{int} V|}}$. Even to use this method for bipartite graphs with 4 interior vertices would require first demonstrating nontrivial labelings for upwards of 1400 graphs.

5 A Combinatorial Interpretation of $\det C$

The following lemma is an elementary result in algebra.

Lemma 5.1. *(Smith Normal Form.) Suppose R is a principal ideal domain and A is an $n \times n$ matrix with entries in R. Then there is a decomposition $A = SDT$ where D is diagonal with entries in R and $\det S$ and $\det T$ are units in R.*

Using this we prove the following Lemma.

Lemma 5.2. *Suppose A is an $n \times n$ matrix with entries in $\mathbb{Z}$ with nonzero determinant, viewed as a map on $\mathbb{Q}$. Let $A_{\mathbb{Q}/\mathbb{Z}}$ be the natural map $A_{\mathbb{Q}/\mathbb{Z}} \colon (\mathbb{Q}/\mathbb{Z})^n \to (\mathbb{Q}/\mathbb{Z})^n$. Then $|\ker A_{\mathbb{Q}/\mathbb{Z}}| = |\det A|$.*

Proof. Using Smith Normal Form, we can write $A = SDT$ for S, D, T matrices with entries in $\mathbb{Z}$, and S, T having determinants ± 1. Then

$$|\det A| = \prod_{i=1}^{n} |D_{ii}|.$$

On the other hand, since S, T have unit determinants they are invertible and thus a vector $x \in \mathbb{Q}/\mathbb{Z}$ is in the kernel of $A_{\mathbb{Q}/\mathbb{Z}}$ if and only if it is in the kernel of D, which is equivalent to having $D_{ii}x_i = 0$ in $\mathbb{Q}/\mathbb{Z}$ for all i. Since $D_i i$ is a nonzero integer, this has D_{ii} solutions for x_i in $\mathbb{Q}/\mathbb{Z}$ and thus there are $\prod |D_{ii}|$ elements of the kernel of $A_{\mathbb{Q}/\mathbb{Z}}$. This gives the result. □

Using the above we prove the following Theorem

Theorem 5.3. *Suppose Γ is a Dirichlet non-singular network. Then the number of $\mathbb{Q}/\mathbb{Z}$ harmonic functions that are 0 on the boundary is equal to $|\det(C)|$.*

Proof. Observe that

$$\ker C_{\mathbb{Q}/\mathbb{Z}} \cong \mathcal{U}_{\Gamma,\mathbb{Q}/\mathbb{Z}}^{0 \text{ on boundary}},$$

since if $u \in \ker C_{\mathbb{Q}/\mathbb{Z}}$ then $(0\ u)$ (where 0 is a row vector filled with $|\partial V|$ 0s), is harmonic as is easily checked. We can do the same thing in reverse for any $v = (0\ u)$ that is $\mathbb{Q}/\mathbb{Z}$ harmonic and 0 on the boundary. From Lemma 5.2 we get that $|\ker C_{\mathbb{Q}/\mathbb{Z}}| = \det C$, and thus we get the result. □

Corollary 5.4. *Suppose Γ is a Dirichlet non-singular network. Then $|\mathcal{U}^1_{\Gamma,\mathbb{Z}}| < \infty$.*

Proof. Given any $u \in \mathcal{U}_{\Gamma,\mathbb{Q}/\mathbb{Z}}$, by the Dirichlet nonsingularity (implying that the Dirichlet problem can be solved over $\mathbb{Q}$), we can find a $\mathbb{Q}$ harmonic function with the same boundary values as u since C is invertible. Call this $\mathbb{Q}$ harmonic function v. Then we can project v onto $\mathbb{Q}/\mathbb{Z}$. We get then that $u - v$ is $\mathbb{Q}/\mathbb{Z}$ harmonic with all boundary values 0. But if we are working in the harmonic cohomology we don't care about the images of $\mathbb{Q}$ harmonic functions, and thus we get that, up to the image of a $\mathbb{Q}$ harmonic function, each $\mathbb{Q}/\mathbb{Z}$ harmonic function is equivalent to one with boundary values 0. It follows from the previous theorem therefore that $|\mathcal{U}^1_{\Gamma,\mathbb{Z}}| \leq |\det C| < \infty$. □

6 Miscellaneous Algebraic Lemmas

In this section, we collect some minor lemmas which were not explored to the same depth as the other topics in this paper. In particular, we have some basic results on extending the derived functor to torsion modules or allowing the conductances to be nonunits.

6.1 Factorization

Lemma 6.1. *Let R be a PID and F its field of fractions. Any F/R-valued harmonic function u can be expressed as $\sum_{j=1}^{n} u_j$, where each $u_j \in \mathcal{U}_{\Gamma,F/R}$ and the denominator of each coordinate of u_j is a power of a prime p_j.*

Proof. For a prime (p), let $T_{(p)}\mathcal{U}_{\Gamma,F/R}$ be the (p)-torsion submodule of $\mathcal{U}_{\Gamma,F/R}$. Then

$$\mathcal{U}_{\Gamma,F/R} \cong \bigoplus_{\text{prime ideals } (p)} T_{(p)}\mathcal{U}_{\Gamma,F/R},$$

because any torsion module over a PID is the direct sum of its p-torsion submodules. In particular, any u can be expressed as $\sum u_j$, where u_j is a p_j-torsion element for non-associate primes. This implies the denominator of each coordinate of u_j is a power of p_j (after multiplying the numerator and denominator by some unit). $\square$

6.2 Quotients by Principal Ideals

While we haven't much investigated the effect of the derived functor on modules other than the ring itself, there are a couple of simple results on quotient modules of PIDs.

Proposition 6.2. *For R a principal ideal domain, F its field of fractions, and $x \in R\backslash\{0\}$, an injective resolution of the ring $R/(x)$ (considered as an R-module) is given by*

$$0 \to R/(x) \xrightarrow{1 \to 1/x} F/R \xrightarrow{\times x} F/R \to 0$$

where the first map is defined by sending 1 to $1/x$ (since 1 generates the module) and the second map is multiplication by x.

Proof. First, F/R is an injective module [3, p. 123]. It remains to show the sequence is exact. The first map is injective: given some $a \in R/(x)$ sent to 0, then $a/x = 0$ in F/R. That is, $a/x \in R$, so a is a multiple of x, and is 0 in $R/(x)$. The kernel of the second map consists of all elements of the form a/x, which is exactly the image of the first map. Finally, any element a/b of F/R can be pulled back under the second map to a/bx. Thus the second map is surjective and the sequence is exact. $\square$

Proposition 6.3. *With R, F, and x as above, $\mathcal{U}^1_{\Gamma,R/(x)} \cong \mathcal{U}_{\Gamma,F/R}/x\mathcal{U}_{\Gamma,F/R}$.*

Proof. This follows from the definition: applying the functor to the injective resolution gives the chain complex

$$0 \to \mathcal{U}_{\Gamma,F/R} \xrightarrow{\times x} \mathcal{U}_{\Gamma,F/R} \to 0$$

Then

$$\mathcal{U}^1_{\Gamma,R/(x)} \cong \frac{\mathcal{U}_{\Gamma,F/R}}{\mathrm{im}(\times x)} \cong \frac{\mathcal{U}_{\Gamma,F/R}}{x\mathcal{U}_{\Gamma,F/R}}$$

□

While the cohomologies for quotient rings seem to have a very different description from that of the ring itself, they turn out to be closely connected.

Theorem 6.4. *For any PID,* $\mathcal{U}^1_{\Gamma,R/(x)} \cong \mathcal{U}^1_{\Gamma,R}/x\mathcal{U}^1_{\Gamma,R}$.

Proof. Consider the short exact sequence $0 \to R \xrightarrow{\times x} R \to R/(x) \to 0$, where the map $R \to R$ is multiplication by x. Since $\mathcal{U}_{\Gamma,-}$ is left exact, we can apply it to our short exact sequence and extend the result to a long exact sequence of derived functors

$$0 \to \mathcal{U}_{\Gamma,R} \xrightarrow{\times x} \mathcal{U}_{\Gamma,R} \to \mathcal{U}_{\Gamma,R/(x)} \to \mathcal{U}^1_{\Gamma,R} \xrightarrow{\times x} \mathcal{U}^1_{\Gamma,R} \twoheadrightarrow \mathcal{U}^1_{\Gamma,R/(x)} \to 0 \to \dots$$

Hence, $\mathcal{U}^1_{\Gamma,R/(x)} \cong \mathcal{U}^1_{\Gamma,R}/\ker(\mathcal{U}^1_{\Gamma,R} \to \mathcal{U}^1_{\Gamma,R/(x)})$, where $\ker(\mathcal{U}^1_{\Gamma,R} \to \mathcal{U}^1_{\Gamma,R/(x)}) \cong \mathrm{im}(\times x) \cong x\mathcal{U}^1_{\Gamma,R}$. □

Corollary 6.5. $\mathcal{U}^1_{\Gamma,\mathbb{Z}/n} \cong \mathcal{U}^1_{\Gamma,\mathbb{Z}}/n\mathcal{U}^1_{\Gamma,\mathbb{Z}}$.

Corollary 6.6. *If* $\mathcal{U}^1_{\Gamma,R}$ *is finite, then* $\mathcal{U}^1_{\Gamma,R} \cong 0$ *if and only if* $\mathcal{U}^1_{\Gamma,R/(x)} \cong 0$ *for every* $x \in R\backslash\{0\}$.

Proof. If $\mathcal{U}^1_{\Gamma,R}$ is trivial, then certainly every quotient of it is trivial. Conversely, if $\mathcal{U}^1_{\Gamma,R}$ is nontrivial, then select some element of $\mathcal{U}_{\Gamma,F/R}$ from a nonzero equivalence class. Each of the values of this function has a denominator in R, so let x be the product of these denominators. Then multiplication by x sends each of the values of this function to an element in R, so it annihilates the element and thus its equivalence class. Then the map $\times x : \mathcal{U}^1_{\Gamma,R} \to \mathcal{U}^1_{\Gamma,R}$ has nontrivial kernel and is therefore not injective. Since $\mathcal{U}^1_{\Gamma,R}$ is assumed to be finite, the map cannot be surjective either, so the quotient $\mathcal{U}^1_{\Gamma,R}/x\mathcal{U}^1_{\Gamma,R} \cong \mathcal{U}^1_{\Gamma,R/(x)}$ is nontrivial. □

Does this mean that it might be easier to consider the groups $\mathcal{U}^1_{\Gamma,R/(x)}$ rather than the whole group $\mathcal{U}^1_{\Gamma,R}$? This is uncertain. However, if it is, the following lemma might help in that regard.

Lemma 6.7. *Suppose* Γ *is a network with conductances in* $\mathbb{Z}$, v *is an interior vertex of* Γ *with no interior neighbors, and the sum of all conductances adjacent to* v *is an integer* j. *Let* Γ' *denote the network where we delete* v. *If* p *is any integer relatively prime to* j *then*

$$\mathcal{U}^1_{\Gamma,\mathbb{Z}/p} \cong \mathcal{U}^1_{\Gamma',\mathbb{Z}/p}$$

Proof. Agree to use the usual injective resolution:

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{x \mapsto x/p} \mathbb{Q}/\mathbb{Z} \xrightarrow{x \mapsto px} \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \longrightarrow 0 \cdots$$

We can get a surjection from

$$\mathcal{U}_{\Gamma,\mathbb{Q}/\mathbb{Z}} \twoheadrightarrow \mathcal{U}_{\Gamma',\mathbb{Q}/\mathbb{Z}}$$

since for any $\mathbb{Q}/\mathbb{Z}$ harmonic function on Γ', we can divide the sum of the voltages at each boundary vertex adjacent to v by j (since we're in $\mathbb{Q}/\mathbb{Z}$) to find a $\mathbb{Q}/\mathbb{Z}$ harmonic function on Γ that has the same values on Γ'. Hence the surjectivity. Letting K_1 and K_2 denote the kernels of the following maps in the rows, we get two rows of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \xhookrightarrow{\text{id}} & \mathcal{U}_{\Gamma,\mathbb{Q}/\mathbb{Z}} & \twoheadrightarrow & \mathcal{U}_{\Gamma',\mathbb{Q}/\mathbb{Z}} & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & K_2 & \xhookrightarrow{\text{id}} & \mathcal{U}_{\Gamma,\mathbb{Q}/\mathbb{Z}} & \twoheadrightarrow & \mathcal{U}_{\Gamma',\mathbb{Q}/\mathbb{Z}} & \longrightarrow & 0 \end{array}$$

Where the f, g, h are the natural projections given by our injective resolution; i.e. the maps where we multiply the elements of the domain by p. But now observe that $K_1 = K_2 \cong \mathbb{Z}/j$ because for a harmonic function u to be in K_1 (or K_2, for that matter) it must be that $ju(v) = 0$ in $\mathbb{Q}/\mathbb{Z}$, and $u(w) = 0$ for the rest of the vertices. This equation has solutions isomorphic to $\mathbb{Z}/j$, and thus the claim. From here we can apply the Snake Lemma to get the short exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0.$$

Next we notice that since p is relatively prime to n, and f is a multiplication by p map, we get that f is invertible, so $\ker f \cong 0$, and $\operatorname{coker} f \cong 0$. Therefore we get that

$$\ker g \cong \ker h \quad \text{and} \quad \operatorname{coker} g \cong \operatorname{coker} h.$$

This second isomorphism is our claim. □

6.3 Multiplying the Conductances by a Constant

While our insistence on conductances being units is necessary to preserve the geometric connection with contracting boundary spikes, there is no reason not to consider what happens in more general cases. While we have not closely examined how this deviates from the unit case, we have the following result in the case that the conductances are all multiples of units by the same constant.

For a network Γ and c a nonzero element of the conductance ring, we denote the network in which all conductances have been multiplied by c by $c\Gamma$.

Theorem 6.8. *If R is a PID, there is an exact sequence*

$$0 \to \mathcal{U}^1_{\Gamma,R} \hookrightarrow \mathcal{U}^1_{c\Gamma,R} \twoheadrightarrow \mathcal{U}^1_{c\Gamma,R/(c)} \to 0$$

Proof. A harmonic function on Γ taking values in any module M is also harmonic on $c\Gamma$; changing from Γ to $c\Gamma$ simply multiplies the equations defining harmonicity by c, so any function satisfying the original equations satisfies the new ones. Furthermore, if the action of multiplication by c on M is injective, then any function satisfying the new equations satisfies the old ones as well; this is true of the field of fractions F, so $\mathcal{U}_{\Gamma,F} \cong \mathcal{U}_{c\Gamma,F}$. Let $Q_{F/R} = \operatorname{coker}(\mathcal{U}_{\Gamma,F/R} \hookrightarrow \mathcal{U}_{c\Gamma,F/R})$; then we have the diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{U}_{\Gamma,F} & \overset{\text{id}}{\hookrightarrow} & \mathcal{U}_{c\Gamma,F} & \twoheadrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h & & \\
0 & \longrightarrow & \mathcal{U}_{\Gamma,F/R} & \overset{\text{id}}{\hookrightarrow} & \mathcal{U}_{c\Gamma,F/R} & \twoheadrightarrow & Q_{F/R} & \longrightarrow & 0
\end{array}$$

Which, by the snake lemma, produces an exact sequence

$$0 \to \mathcal{U}_{\Gamma,R} \to \mathcal{U}_{c\Gamma,R} \to 0 \to \mathcal{U}^1_{\Gamma,R} \to \mathcal{U}^1_{c\Gamma,R} \to Q_{F/R} \to 0$$

It remains to determine $Q_{F/R}$. Note that $\mathcal{U}_{\Gamma,F/R} \cong \ker \left.\begin{pmatrix} B^T & C \end{pmatrix}\right|_{F/R}$ while $\mathcal{U}_{c\Gamma,F/R} \cong \ker c \left.\begin{pmatrix} B^T & C \end{pmatrix}\right|_{F/R}$; then we claim that $\mathcal{U}_{\Gamma,F/R} = c\mathcal{U}_{c\Gamma,F/R}$. Given some $u \in \mathcal{U}_{\Gamma,F/R}$, we can consider some function $v = u/c$ (since we are working in F/R, this exists, although not uniquely), and then $c \begin{pmatrix} B^T & C \end{pmatrix} v = \begin{pmatrix} B^T & C \end{pmatrix} u = 0$, so $u \in c\mathcal{U}_{c\Gamma,F/R}$. Conversely, given $u = cv$ with $v \in \mathcal{U}_{c\Gamma,F/R}$, then $\begin{pmatrix} B^T & C \end{pmatrix} u = c \begin{pmatrix} B^T & C \end{pmatrix} v = 0$, so $u \in \mathcal{U}_{\Gamma,F/R}$.

With this in place, we have $Q_{F/R} = \mathcal{U}_{c\Gamma,F/R}/\mathcal{U}_{\Gamma,F/R} = \mathcal{U}_{c\Gamma,F/R}/c\mathcal{U}_{c\Gamma,F/R} = \mathcal{U}^1_{c\Gamma,R/(c)}$ by Proposition 6.3, which gives the result.

□

Corollary 6.9. *If $\mathcal{U}^1_{\Gamma,R}$ is nontrivial, so is $\mathcal{U}^1_{c\Gamma,R}$.*

7 Questions that Remain

- The Big Conjecture: Is there a ring such that, if the cohomologies corresponding to all assignments of unit conductances to a bgraph are all trivial, then the graph is quasi-layerable? Is it $\mathbb{Z}[\zeta_3]$?
- Specifically, does $\mathbb{Z}$ suffice? If it doesn't, what's a counterexample bgraph, and can we classify the counterexamples? This leads into The Bigger Question.
- The Bigger Question: Can we in any way classify bgraphs based on the collection of harmonic cohomologies associated to their networks in some ring? In any ring?

- In general, what effect does gluing together graphs at multiple boundary vertices have on the cohomology?
- What effect does changing an interior vertex to boundary have on the cohomology if it does have interior neighbors?
- How do these results extend to non-PIDs?
- How do these results extend to non-unit conductances?
- Is discrete harmonic cohomology always finite/rank 0?
- Is $(B^T C)_F$ not onto exactly when the graph is layerable? If so then we can characterize the times when the cokernel interpretation fails to be trivial anyways.
- What are the algebraic properties & cohomologies of Avi's quad-graphs?
- Which non-quasi-layerable graphs have no proper non-quasi-layerable subbgraphs?
- Dirichlet non-singularity is sufficient to give a finite cohomology, but it's not necessary. So what is a necessary condition for finite cohomology?

OTHER IDEAS:

- What happens when you multiply all the conductances by a constant.
- Folding duplicate boundary vertices together.
- Reducing series connections which are Dirichlet-singular.
- For a PID, $\mathcal{U}^1_{\Gamma,R/(x)} \cong \mathcal{U}^1_{\Gamma,R}/x\mathcal{U}^1_{\Gamma,R}$.
- What does it mean for two networks to have isomorphic first cohomology modules? Is this a good classification scheme for networks?? Maybe analogies with non-simply connected surfaces will tell us this is hopeless...
- Here is a more concentrated version of the above question: suppose Γ and Γ' are two bipartite networks with all conductances 1, and $\mathcal{U}^1_{\Gamma,\mathbb{Z}} \cong \mathcal{U}^1_{\Gamma',\mathbb{Z}}$s. What can we say about the underlying bgraphs? Perhaps some sort of topological isomorphism? I don't know enough examples of networks with the same first cohomology module to know whether this is a dumb question or not.
- Other examples–include as needed.

REFERENCES–cite some standard papers on electrical networks. Cite standard algebra textbooks if you think it's necessary. May be useful to future students reading the paper. Also, cite Lam and Pylyavsky for thinking of studying the multiple n-gon graphs.

DIAGRAMS–use tikz or learn how to use TikzCD.

Plan for organization:

1. Definitions
   (a) Category of networks (including basic network definitions such as layerability)
   (b) $\mathcal{U}_{-,-}$ as a bifunctor, left-exactness
   (c) necessary homological algebra
2. Introducing $\mathcal{U}^1_{\Gamma,M}$:
   (a) What it is for a PID
   (b) coker$(B^T\,C)_{\mathbb{Z}}$
   (c) Brute-force calculation of $K_{m,n}$? and $K_{m,2}$ with the interior edge
3. Transforming Graphs
   (a) Implications of layerability
   (b) The converse conjecture (should this go at the end, or the beginning?)
   (c) Pasting
   (d) Star-pasting & turning interior vertices into boundary
   (e) Constructing all finite abelian groups
4. Assigning unit conductances to make $\mathcal{U}^1_{\Gamma,\mathbb{Z}[\zeta_3]}$ nontrivial
   (a) On bipartite graphs
   (b) $|intV| \geqslant |\partial V|$
   (c) strategies for $|\partial V| > |intV|$.
5. $\mathbb{Q}/\mathbb{Z}$-valued harmonic functions 0 on boundary
   (a) det C and spanning trees
   (b) more on the structure from Smith Normal Form?
6. Algebra Aside, or the pack of uselessness
   (a) $\mathcal{U}^1_{\Gamma,R/(x)}$, $\mathcal{U}^1_{\Gamma,R_p}$ effects of deleting interior on $\mathcal{U}^1_{\Gamma,R/(x)}$
   (b) multiplying conductances by constant

A Terms from Homological Algebra

In this section, we will present a brief introduction to the homological algebra necessary to understand these results. We will not present proofs, and refer readers to Vermani [3] or any other introduction to the subject.

A.1 Exact Sequences

Let R be a ring. Then consider a sequence of R-modules and homomorphisms between them:

$$\cdots \longrightarrow M_{i-1} \xrightarrow{\phi_i} M_i \xrightarrow{\phi_{i+1}} M_{i+1} \xrightarrow{\phi_{i+2}} M_{i+2} \longrightarrow \cdots$$

Such a sequence is called a **cochain complex** if for all k, $\phi_{k+1} \circ \phi_k = 0$, or equivalently, $\operatorname{im}(\phi_k) \subseteq \ker(\phi_{k+1})$. In the special case that $\operatorname{im}(\phi_k) = \ker(\phi_{k+1})$, the sequence is called **exact**.

Given a cochain complex like the above (denoted $M^\bullet$), its **cohomology modules** are defined by

$$H_k(M^\bullet) = \frac{\ker(\phi_{k+1})}{\operatorname{im}(\phi_k)}$$

Note that the sequence is exact if and only if the cohomology modules are all trivial. In a sense, the cohomology measures how much the sequence fails to be exact.

A **short exact sequence** is one of the form

$$0 \longrightarrow L \xhookrightarrow{\phi} M \xrightarrow{\psi} N \longrightarrow 0$$

The exactness packages together the information that:

- The image of the first map is 0, so $\ker(\phi) = 0$. Thus ϕ is injective.
- The kernel of the last map is N, so $\operatorname{im}(\psi) = N$. Thus ψ is surjective.
- $\ker(\psi) = \operatorname{im}(\phi)$. By the first isomorphism theorem, $N \cong M/\operatorname{im}(\phi)$. If the injection ϕ is taken as identifying L with a submodule of M, $N \cong M/L$.

Note that this last point gives us a way of extending any injection or surjection into an exact sequence, to which we can apply the techniques of homological algebra:

- Given an injection $\iota : L \hookrightarrow M$, inserting $\operatorname{coker}(\iota) = M/\operatorname{im}(\iota)$ gives the exact sequence $0 \to L \hookrightarrow M \twoheadrightarrow \operatorname{coker}(\iota) \to 0$.
- Given a surjection $\pi : M \hookrightarrow N$, inserting $\ker(\pi)$ gives the exact sequence $0 \to \ker(\pi) \hookrightarrow M \twoheadrightarrow N \to 0$.

Given a functor $\mathcal{F} : \textbf{R-Mod} \to \textbf{S-Mod}$ for rings R and S, if $\phi_{k+1} \circ \phi_k = 0$, $\mathcal{F}(\phi_{k+1}) \circ \mathcal{F}(\phi_k) = \mathcal{F}(0) = 0$, so applying the functor to a cochain complex produces a cochain complex. We can then ask whether a functor preserves short exact sequences. The answer is "sometimes, partially".

We say a $\mathcal{F}$ is an **exact functor** if for every short exact sequence

$$0 \longrightarrow L \overset{\phi}{\hookrightarrow} M \overset{\psi}{\twoheadrightarrow} N \longrightarrow 0$$

the sequence

$$0 \longrightarrow \mathcal{F}(L) \overset{\mathcal{F}(\phi)}{\hookrightarrow} \mathcal{F}(M) \overset{\mathcal{F}(\psi)}{\twoheadrightarrow} \mathcal{F}(N) \longrightarrow 0$$

is exact. Similarly, $\mathcal{F}$ is a **left exact functor** if only the sequence

$$0 \longrightarrow \mathcal{F}(L) \overset{\mathcal{F}(\phi)}{\hookrightarrow} \mathcal{F}(M) \xrightarrow{\mathcal{F}(\psi)} \mathcal{F}(N)$$

is exact.

A.2 Derived Functors

Given a left exact functor, there is a canonical way to extend the truncated exact sequence obtained by applying it to an exact sequence into a longer one, with the right derived functors.

First, an **injective** module I is one such that, given any diagram of the following type:

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & \overset{\exists}{\swarrow} & \\ I & & \end{array}$$

there exists a map $B \to I$ which makes the diagram commute. Intuitively, any map from a module A into I can be extended to take values on any supermodule of A. Another way of stating this property is that the functor $\hom(-, I)$, which takes a module and produces the module of all homomorphisms from it to I, is exact.

Then, given an arbitrary module M, an **injective resolution** of M is an exact sequence

$$0 \longrightarrow M \hookrightarrow N^0 \longrightarrow N^1 \longrightarrow N^2 \longrightarrow \cdots$$

such that the modules N^j are injective.

Proposition A.1 ([3, p. 123])**.** *Every module has an injective resolution.*

The rings we deal with in this paper are usually principal ideal domains, which have simple injective resolutions [3, p. 123]:

Proposition A.2. *For any principal ideal domain R, let F be its field of fractions, and consider R and F as R-modules. Then*

$$0 \longrightarrow R \longrightarrow F \longrightarrow F/R \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

is an injective resolution of R.

For example, $\mathbb{Z}$ has the injective resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.

With these definitions in hand in hand, we can now define derived functors. Given a left exact functor $\mathcal{F}$ and an injective resolution of some module $0 \to M \to N^0 \to N^1 \to \ldots$, consider the cochain complex $\mathcal{F}(N^\bullet)$ given by

$$0 \longrightarrow \mathcal{F}(N^0) \longrightarrow \mathcal{F}(N^1) \longrightarrow \mathcal{F}(N^2) \longrightarrow \cdots$$

(Note that M has been dropped from the sequence. The result is still a cochain complex because the composition of the first two maps is trivially 0.) Then the **kth right derived functor** at M is the kth cohomology module of the sequence:

$$\mathcal{F}^k(M) = H^k(\mathcal{F}(N^\bullet))$$

Under this definition, the zeroth derived functor reduces to the original functor itself [3, p. 142]:

Theorem A.3. *For $\mathcal{F}$ a left exact functor, $\mathcal{F}^0$ is naturally equivalent to $\mathcal{F}$. In particular, for any module M, $\mathcal{F}^0(M) \cong \mathcal{F}(M)$.*

The value of right derived functors is that they can be used to extend left exact sequences [3, p. 140]:

Theorem A.4. *If $0 \to L \to M \to N \to 0$ is an exact sequence and $\mathcal{F}$ is a left exact functor, then there is a long exact sequence*

$$0 \longrightarrow \mathcal{F}(L) \longrightarrow \mathcal{F}(M) \longrightarrow \mathcal{F}(N) \longrightarrow \mathcal{F}^1(L)$$

$$\longrightarrow \mathcal{F}^1(M) \longrightarrow \mathcal{F}^1(N) \longrightarrow \mathcal{F}^2(L) \longrightarrow \cdots$$

A.3 The Snake Lemma

Lemma A.5 ([3, p. 101])**.** *Consider the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

such that each of its rows is exact. Then there is an exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker g \longrightarrow \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \longrightarrow 0$$

with the maps between kernels and the maps between cokernels induced by the corresponding maps in the diagram.

The Snake Lemma is a special case of a more general result, which we will not refer to as the "Really Long Snake Lemma".

Theorem A.6 ([3, p. 110])**.** *Given a commutative diagram*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^0 & \longrightarrow & M^0 & \longrightarrow & N^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^1 & \longrightarrow & M^1 & \longrightarrow & N^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^2 & \longrightarrow & M^2 & \longrightarrow & N^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

whose rows are exact and whose columns are cochain complexes, there is an exact sequence

$$0 \longrightarrow H^0(L^\bullet) \longrightarrow H^0(M^\bullet) \longrightarrow H^0(N^\bullet) \longrightarrow H^1(L^\bullet)$$

$$\longrightarrow H^1(M^\bullet) \longrightarrow H^1(N^\bullet) \longrightarrow H^2(L^\bullet) \longrightarrow \cdots$$

As mentioned above, we will mostly be considering injective resolutions with only 3 terms, which produce cochain complexes with only 2 terms. In this case, the Snake Lemma is all we need.

References

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