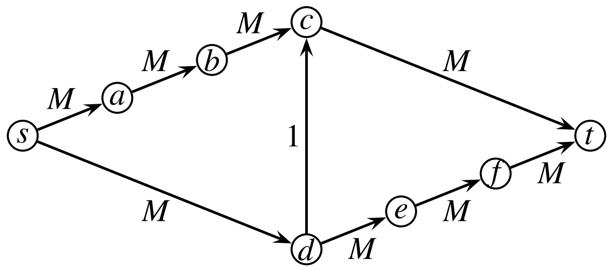
Problem Set 4

409 - Discrete Optimization

Spring 2022

Exercise 1 (6 points)
Consider the following network with a directed graph $G = (V, E)$, capacities $u(e)$ (the labels of the edges), a source s and a sink t (assume that $M > 1$).



a) Argue that the Ford-Fulkerson algorithm with a poor choice of augmenting paths might take M or more iterations.

b) Run the Edmonds-Karp algorithm on this network and give the flow in each iteration.

Exercise 2 (7 points)
Let (G, u, s, t) be a network with $G = (V, E)$[1] and let $f : E \to \mathbb{R}_{\geq 0}$ be a feasible s-t flow in that network. Moreover let f^* be a maximum s-t flow.

a) Let $S \subseteq V$ with $s \in S, t \notin S$. Prove that

$$\text{value}(f^*) \leq \text{value}(f) + \sum_{e \in \delta^+_{E_f}(S)} u_f(e)$$

where $\delta^+_{E_f}(S) := \{(i, j) \in E_f \mid i \in S, j \notin S\}$.

b) Let $\gamma > 0$ be the maximum value so that there is an s-t path P in G_f with $u_f(e) \geq \gamma$ for all $e \in P$. Prove that $\text{value}(f^*) \leq \text{value}(f) + \gamma m$ where $m := |E|$.

Exercise 3 (7 points)
Let (G, u, s, t) be a network with $n = |V|$ nodes and $m = |E|$ edges and $u(e) \in \mathbb{Z}_{\geq 0}$ for all $e \in E$. Suppose that f^* is a maximum s-t flow. Consider the following algorithm (which is a smarter version of Ford-Fulkerson):

[1] As in the lecture notes we make the assumption that G does not contain both edges (i, j) and (j, i) to keep notation clean. Same assumption for exercise 3.

(1) Set $f(e) := 0$ for all $e \in E$
(2) WHILE $\exists$ s-t path in G_f DO

  (3) Compute the s-t path P in G that maximizes $\gamma := \min\{u_f(e) \mid e \in P\}$
  (4) Augment f along P by γ

Note that in each iteration the algorithm chooses the path P that maximizes the *bottleneck capacity*. Let $f_0, f_1, \ldots, f_T$ be the sequence of flows computed by the algorithm where $0 = \text{value}(f_0) < \text{value}(f_1) < \ldots < \text{value}(f_T) = \text{value}(f^*)$.
Hint: For (a) and (b) make use of the previous exercise.

a) Prove that $\text{value}(f_1) \geq \frac{1}{m}\text{value}(f^*)$.

b) Prove that $\text{value}(f_t) \geq \text{value}(f_{t-1}) + \frac{1}{m}(\text{value}(f^*) - \text{value}(f_{t-1}))$ for any $t \geq 1$.

c) Prove that $\text{value}(f_t) \geq \text{value}(f^*) \cdot (1 - (1 - \frac{1}{m})^t)$ for any $t \geq 1$.

d) Prove that the algorithm terminates after $T \leq \lceil m \cdot \ln(2 \cdot \text{value}(f^*)) \rceil$ iterations.