Lecturer: Thomas Rothvoss — Due date: Friday, Oct 27, 2023, 11pm, on GradeScope

Problem Set 4

514 - Networks and Combinatorial Optimization

Autumn 2023

Exercise 4.1 (10pts)
Prove that in a matrix, the maximum number of non-zero entries with no two in the same line (=row or column), is equal to the minimum number of lines that include all nonzero entries.

Example: Consider the following matrix

$$A = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & 0 & 0 \end{pmatrix}$$

where $*$ means any non-zero entry. Then the non-zero entries can be covered by two lines (2nd row and first column) and this is optimal. Also we can select at most 2 non-zero entries that have all rows and columns distinct — for example the two entries (1,1) and (2,2) on the diagonal.

Exercise 4.2 (10pts)
Let $\mathcal{A} = (A_1, \dots, A_n)$ be a family of subsets of some finite set X. Prove that $\mathcal{A}$ has an SDR if and only if

$$\Big|\bigcup_{i \in I} A_i\Big| \geq |I|$$

for each subset $I \subseteq \{1, \dots, n\}$.
Remark: Recall that an SDR is an injective map $\pi : [n] \to X$ with $\pi(i) \in A_i$ for all $i = 1, \dots, n$.

Exercise 4.3 (10pts)
A matrix is called doubly-stochastic if it is nonnegative and each row sum and each column sum is equal to 1. A matrix is palled a permutation matrix if each entry is 0 or 1 and each column and each row contains exactly one 1.

i) Show that for each doubly stochastic matrix $A = (a_{ij})_{i,j=1,\dots,n}$, there exists a permutation $\pi \in S_n$ so that $a_{i,\pi(i)} \neq 0$ for all $i = 1, \dots, n$.

ii) Derive that each doubly stochastic matrix is a convex linear combination of permutation matrices.

Hint: Set up a bipartite graph and prove the claim using König's Theorem.

Remark. All three exercises are taken from A. Schrijver's lecture notes.