Lecturer: Thomas Rothvoss Due date: Friday, Dec 1, 2023, 11pm, on GradeScope

Problem Set 8

514 - Networks and Combinatorial Optimization

Autumn 2023

Exercise 6.4 (10pts)
First answer the following:

(i) Let $D' = (V, A')$ be a directed graph with capacities u' and $s, t \in V$. We call an s-t flow $g : A' \to \mathbb{R}_{\geq 0}$ *elementary* if there is a single s-t path P in D' so that

$$g(a) = \begin{cases} \text{value}(g) & \text{if } a \in P \\ 0 & \text{if } a \notin P \end{cases}$$

Now let f^* be a maximum s-t flow in D' under u'. Prove that there is an elementary s-t flow g under u' with $\text{value}(g) \geq \frac{1}{|A'|}\text{value}(f^*)$.

Now let $D = (V, A)$ be a directed graph with integral capacities $u : A \to \mathbb{Z}_{\geq 0}$, distinguished vertices $s, t \in V$ and for the sake of simplicity suppose that for each arc $a \in A$ one has $a^{-1} \notin A$.

(ii) Let f be any flow under u and let f^* be a maximum value s-t flow. Prove that the residual graph D_f contains an s-t path P where $u_f(a) \geq \frac{1}{2|A|}(\text{value}(f^*) - \text{value}(f))$ for all $a \in P$.

(iii) Consider the modification of the Ford-Fulkerson algorithm where in each iteration we pick an s-t path P that maximizes $\min\{u_f(a) : a \in P\}$ where f is the current flow. Prove that this algorithm takes at most $O(|A| \ln(2\text{value}(f^*)))$ many iterations.

Exercise 6.5 (10pts)
Let $D = (V, A)$ be a directed graph with capacities $u(a) := 1$ for all $a \in A$. Let $s, t \in V$ and assume that $\delta^{\text{out}}(t) = \emptyset$. Let $\delta^{\text{in}}(t) = \{a_1, \ldots, a_m\}$ be the arcs incoming to t. Define $M = (X, \mathcal{I})$ with $X := \{a_1, \ldots, a_m\}$ and $\mathcal{I} := \{\{a_i \in X : f(a_i) = 1\} \mid f \text{ is an } s\text{-}t \text{ flow under } u \text{ in } D\}$. Prove that M is a matroid!