

The Entropy Rounding Method in Approximation Algorithms

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Massachusetts
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Technology



Alexander von Humboldt
Stiftung / Foundation

A general LP rounding problem

Problem:

- ▶ **Given:** $A \in \mathbb{R}^{n \times m}$, fractional solution $x \in [0, 1]^m$
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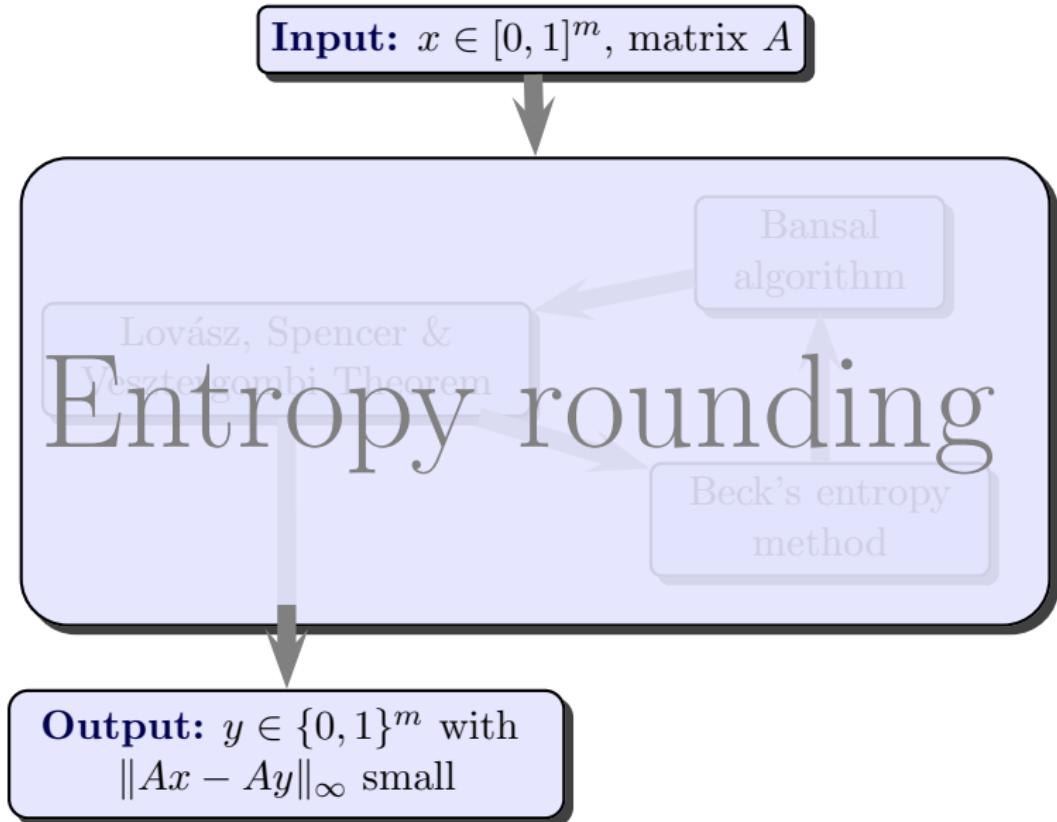
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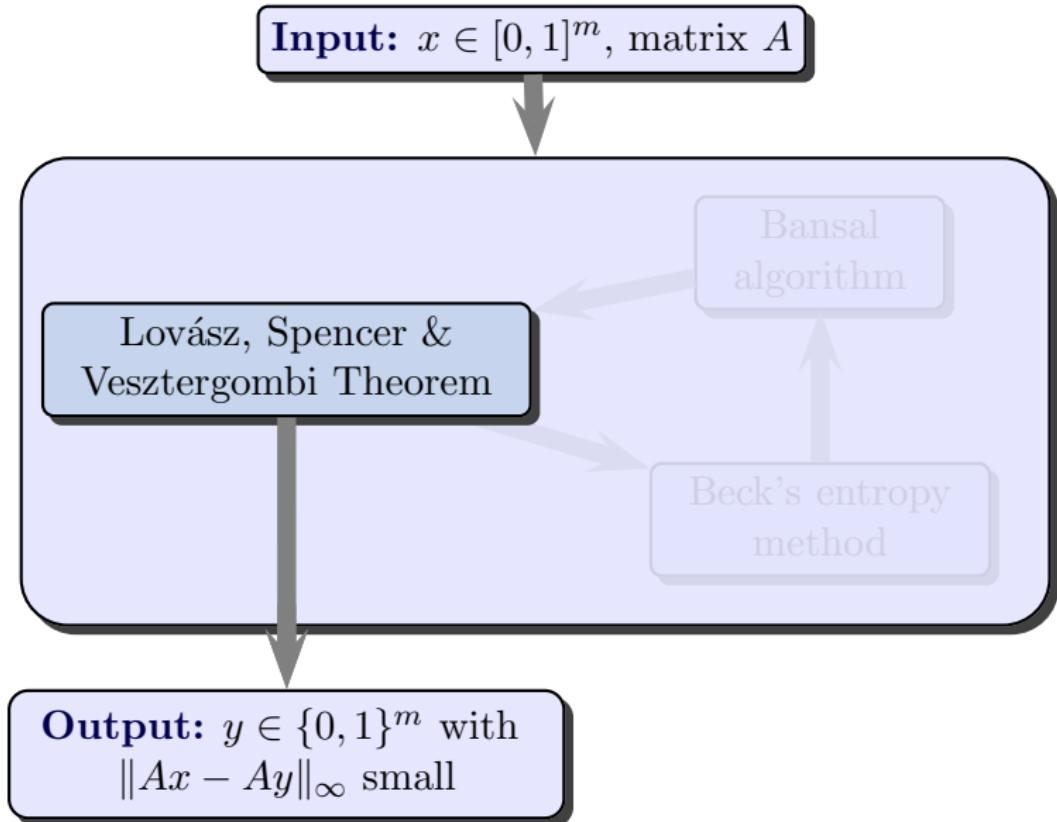
Try another way:

- ▶ “Entropy rounding method” based on discrepancy theory

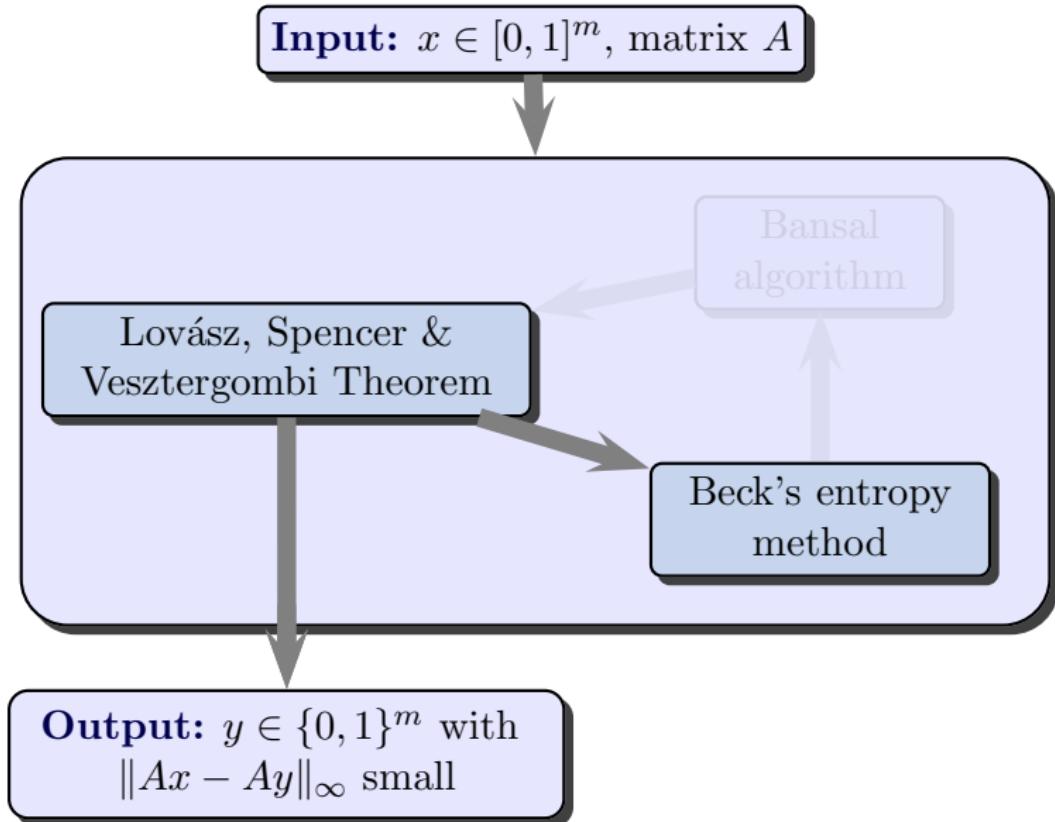
A schematic view



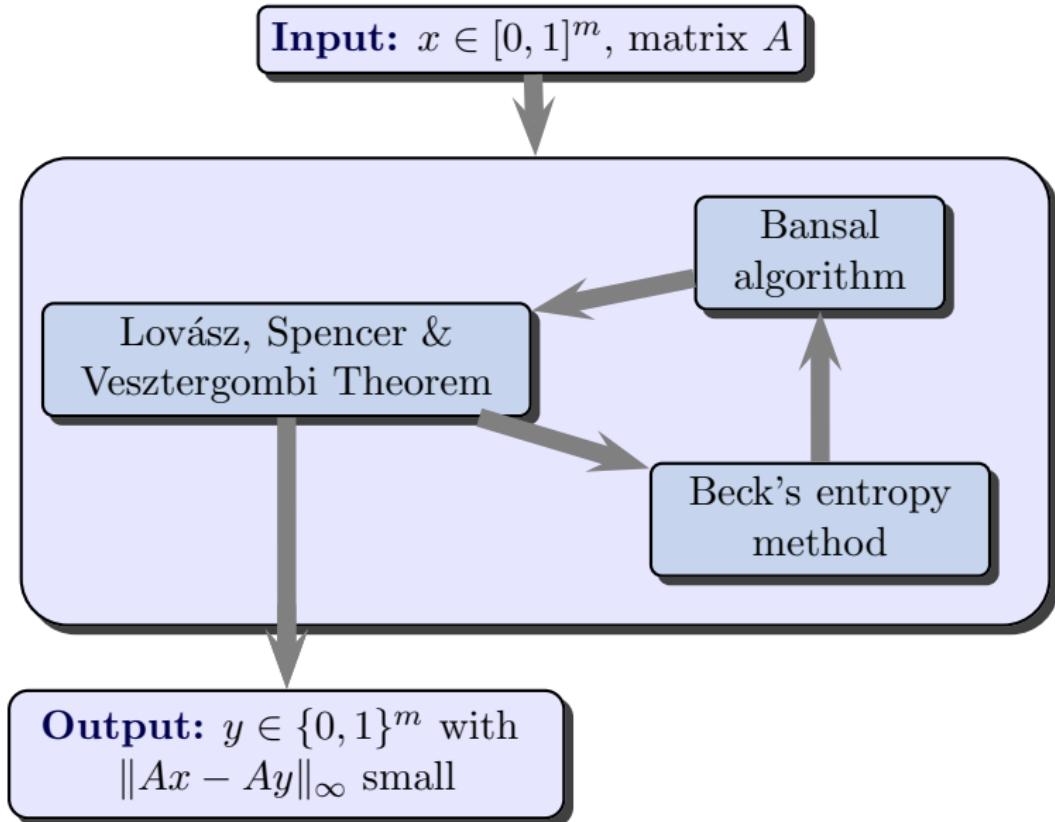
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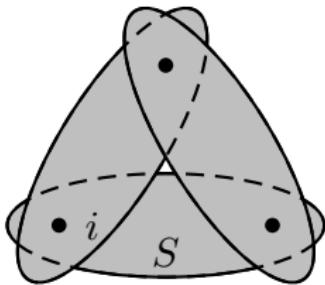


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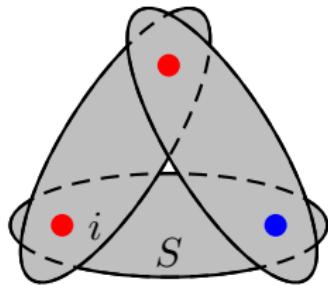
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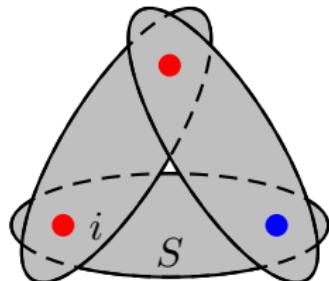


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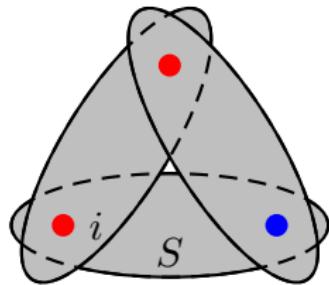
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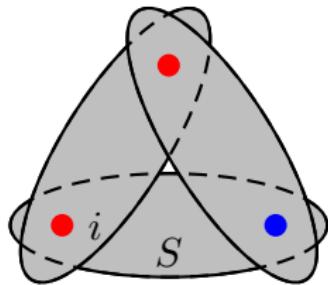
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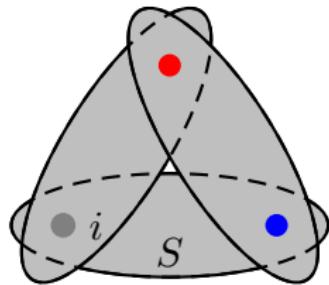
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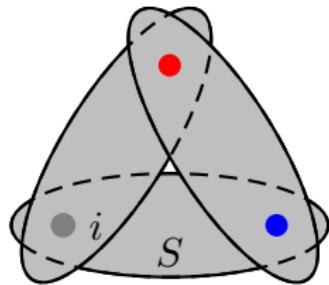
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- ▶ For matrix A : $\text{disc}(A) = \min_{\chi \in \{\pm 1\}^n} \|A\chi\|_\infty$

The LSV-Theorem

Theorem (Lovász, Spencer & Vesztergombi '86)

Given $A \in \mathbb{R}^{n \times m}$, $x \in [0, 1]^m$.

Suppose for any $A' \subseteq A$, \exists coloring $\chi : \|A'\chi\|_\infty \leq \Delta$.
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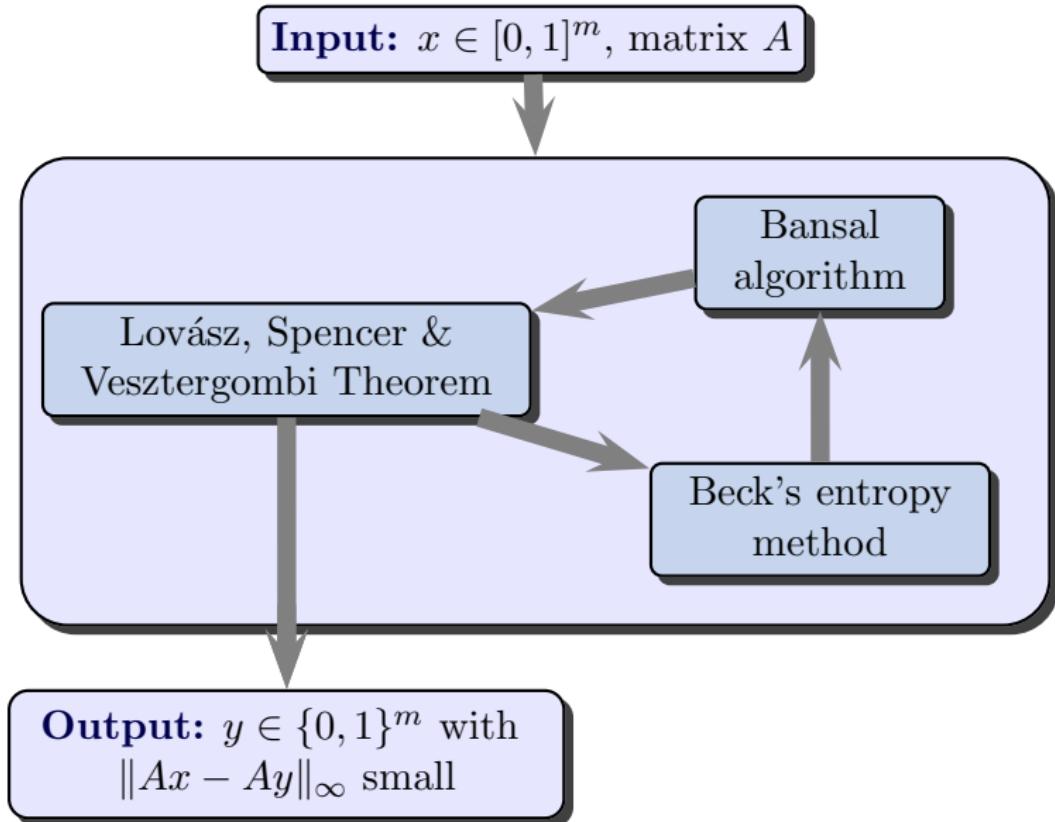
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Definition

For random variable Z , the **entropy** is

$$H(Z) = \sum_z \Pr[Z = z] \cdot \log_2 \left(\frac{1}{\Pr[Z = z]} \right)$$

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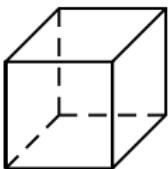
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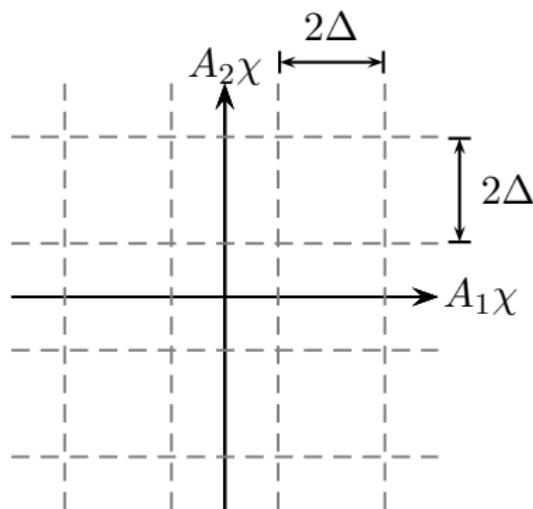
- ▶ *One likely event:* $\exists z : \Pr[Z = z] \geq (\frac{1}{2})^{H(Z)}$
- ▶ *Subadditivity:* $H(f(Z, Z')) \leq H(Z) + H(Z')$.

Theorem [Beck's entropy method]

$$\underset{\chi_i \in \{\pm 1\}}{H} \left(\left\lceil \frac{A\chi}{2\Delta} \right\rceil \right) \leq \frac{m}{5} \Rightarrow \exists \text{half-coloring } \chi^0 : \|A\chi^0\|_\infty \leq \Delta.$$

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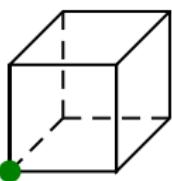
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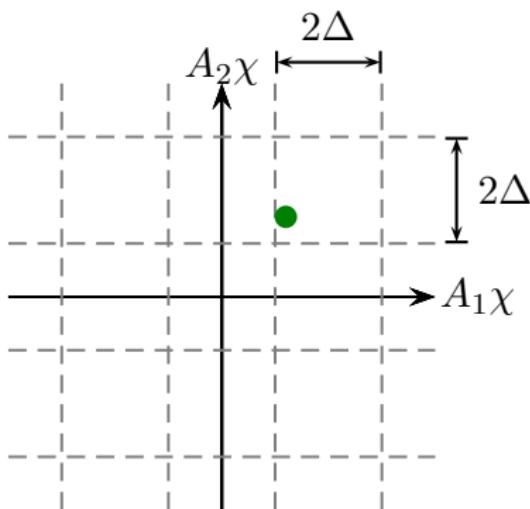


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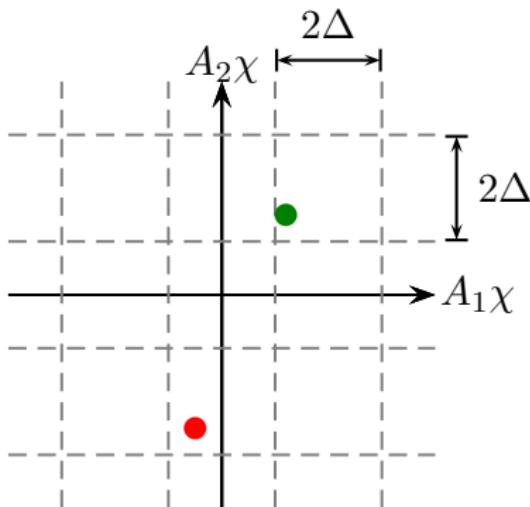
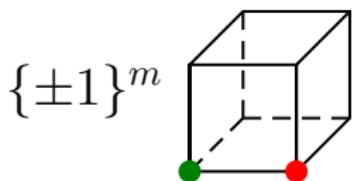
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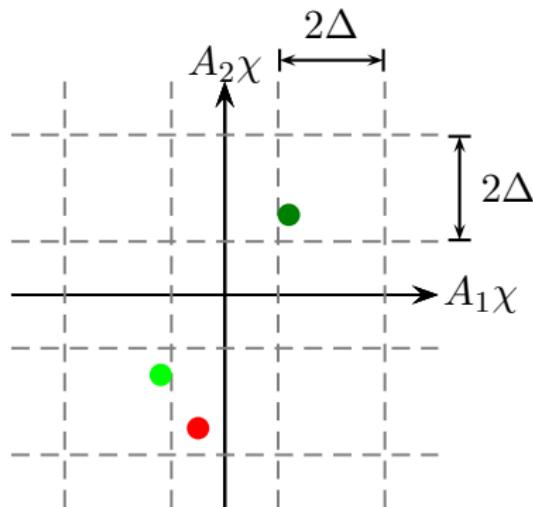
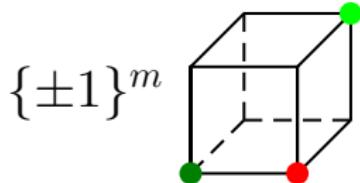
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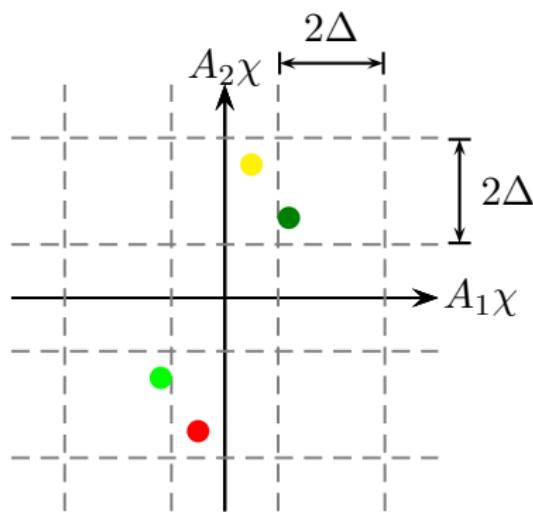
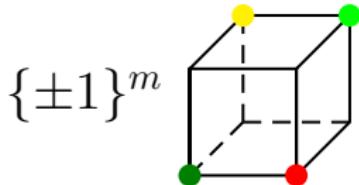
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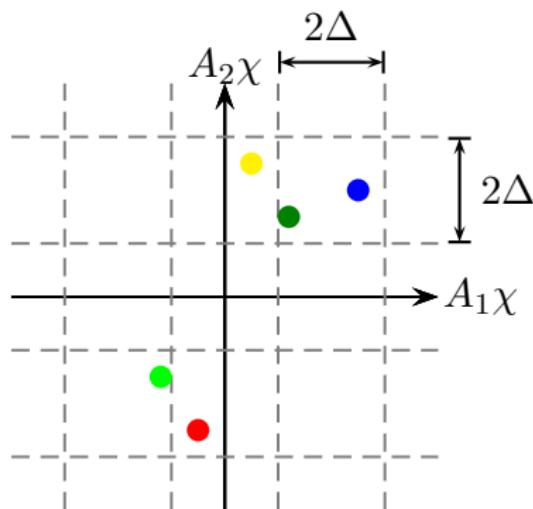
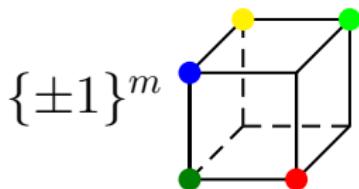
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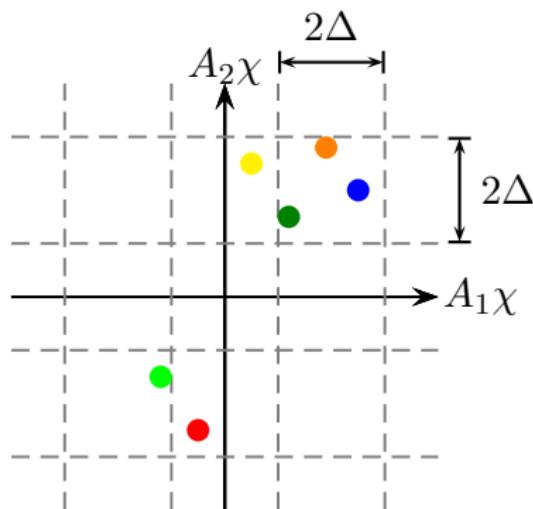
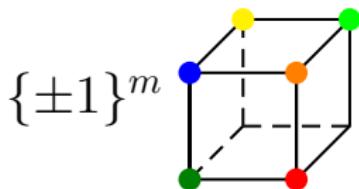
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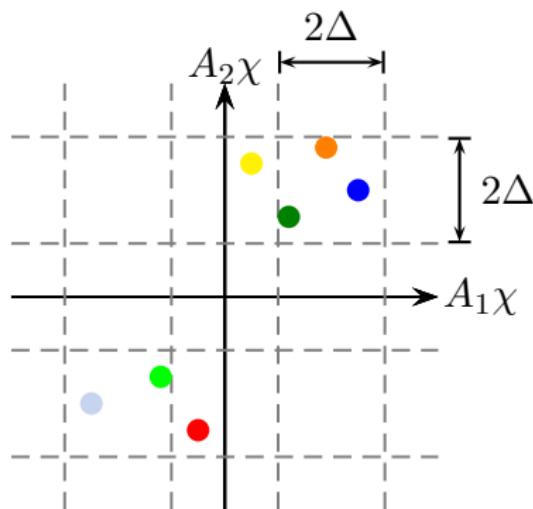
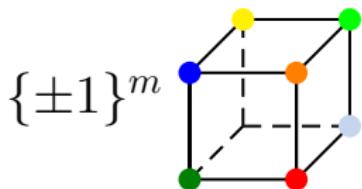
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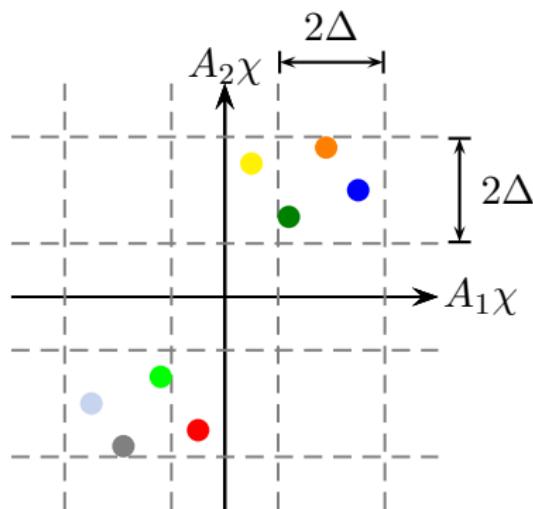
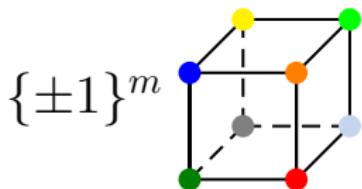
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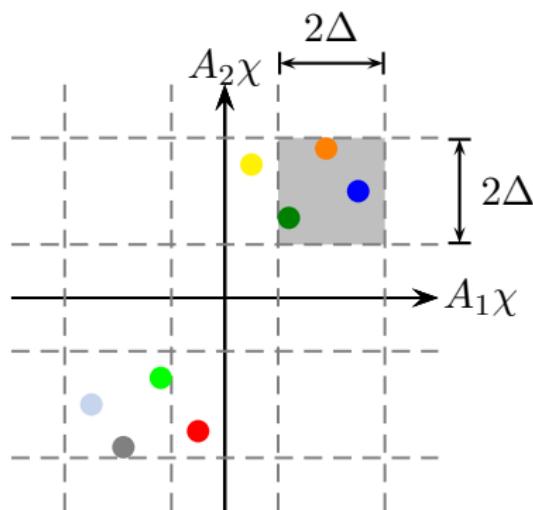
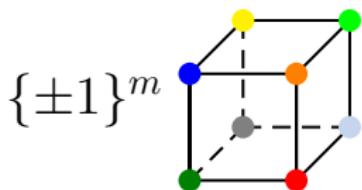


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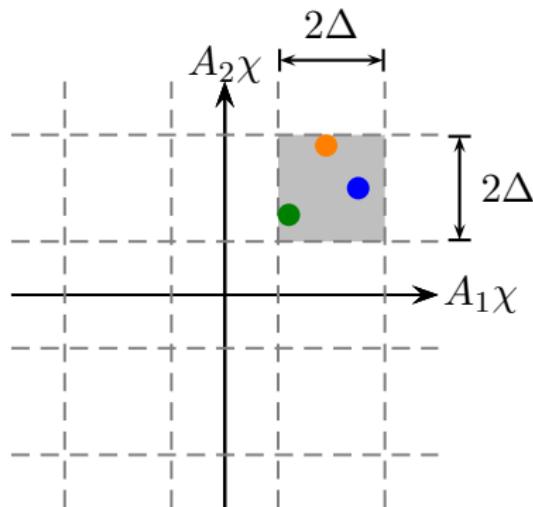
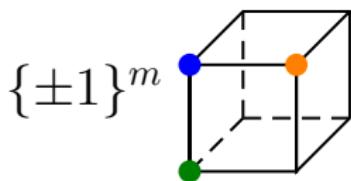


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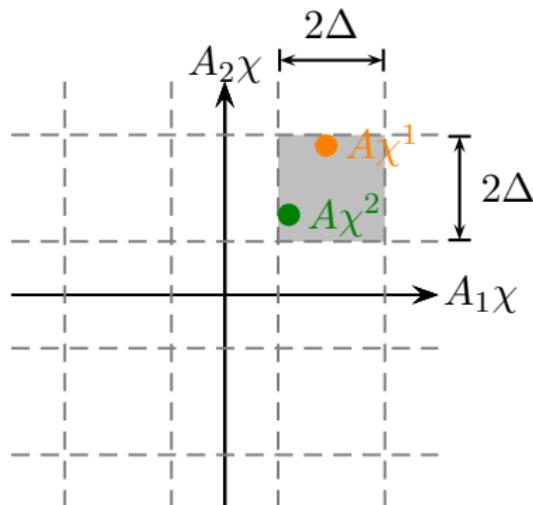
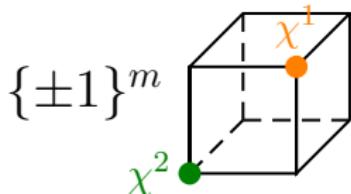


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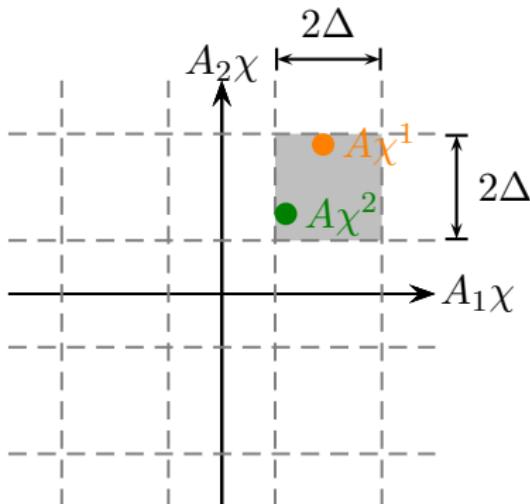
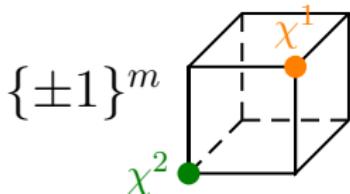


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- ▶ Define $\chi^0(i) := \frac{1}{2}(\chi^1(i) - \chi^2(i)) \in \{0, \pm 1\}.$

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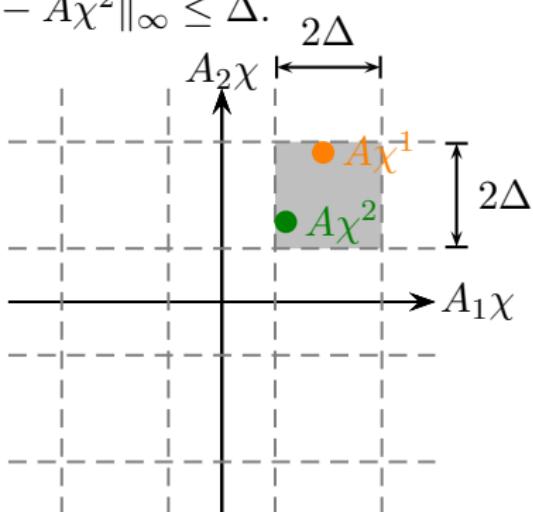
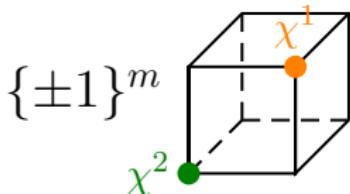


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$$\underset{\chi_i \in \{\pm 1\}}{H} \left(\left\lceil \frac{A\chi}{2\Delta} \right\rceil \right) \leq \frac{m}{5} \Rightarrow \exists \text{half-coloring } \chi^0 : \|A\chi^0\|_\infty \leq \Delta.$$

- ▶ \exists cell : $\Pr[A\chi \in \text{cell}] \geq (\frac{1}{2})^{m/5}$.
- ▶ At least $2^m \cdot (\frac{1}{2})^{m/5} = 2^{0.8m}$ colorings χ have $A\chi \in \text{cell}$
- ▶ Pick χ^1, χ^2 differing in half of entries
- ▶ Define $\chi^0(i) := \frac{1}{2}(\chi^1(i) - \chi^2(i)) \in \{0, \pm 1\}$.
- ▶ Then $\|A\chi^0\|_\infty \leq \frac{1}{2}\|A\chi^1 - A\chi^2\|_\infty \leq \Delta$. □

$$A = \left(\begin{array}{ccc} & m & \\ \hline & \boxed{} & \\ \hline & 3 & \end{array} \right)$$

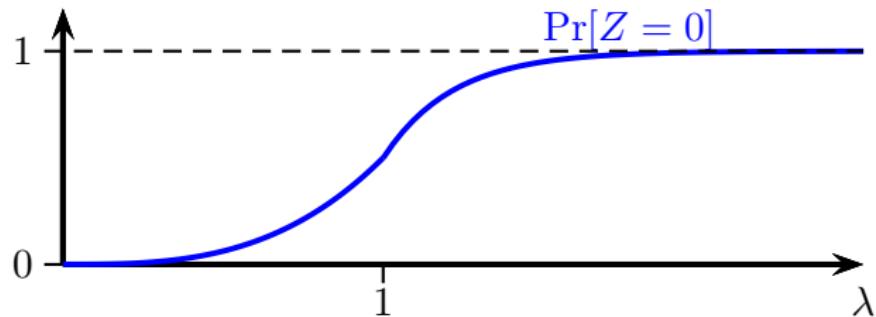


A bound on the entropy

- ▶ Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$.
- ▶ Consider $Z := \left\lceil \frac{\sum_j \chi(j) \cdot \alpha_j}{\lambda \cdot \|\alpha\|_2} \right\rceil$ with $\chi(j) \in \{\pm 1\}$ at random

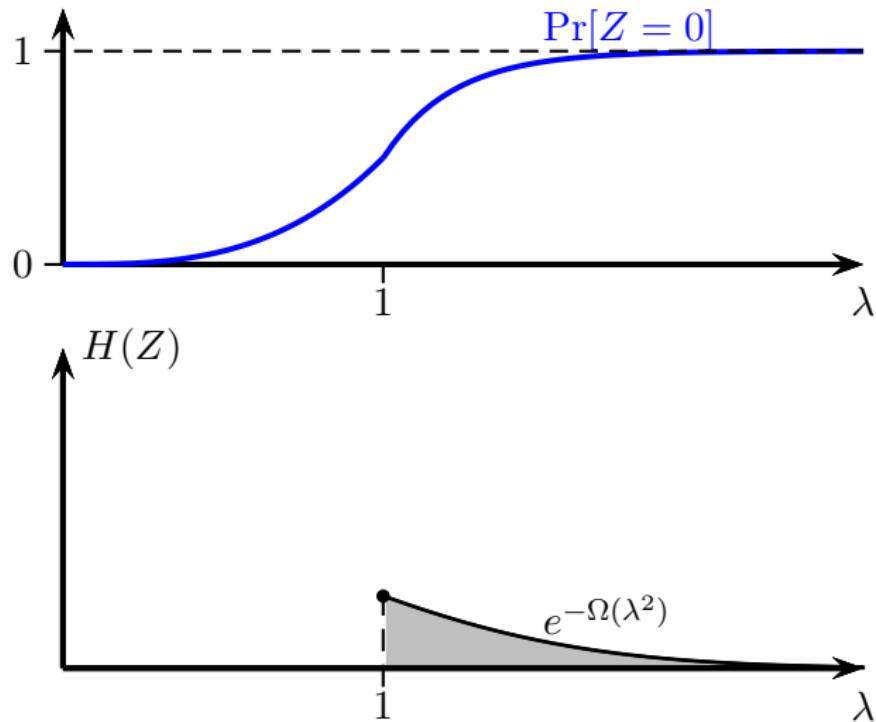
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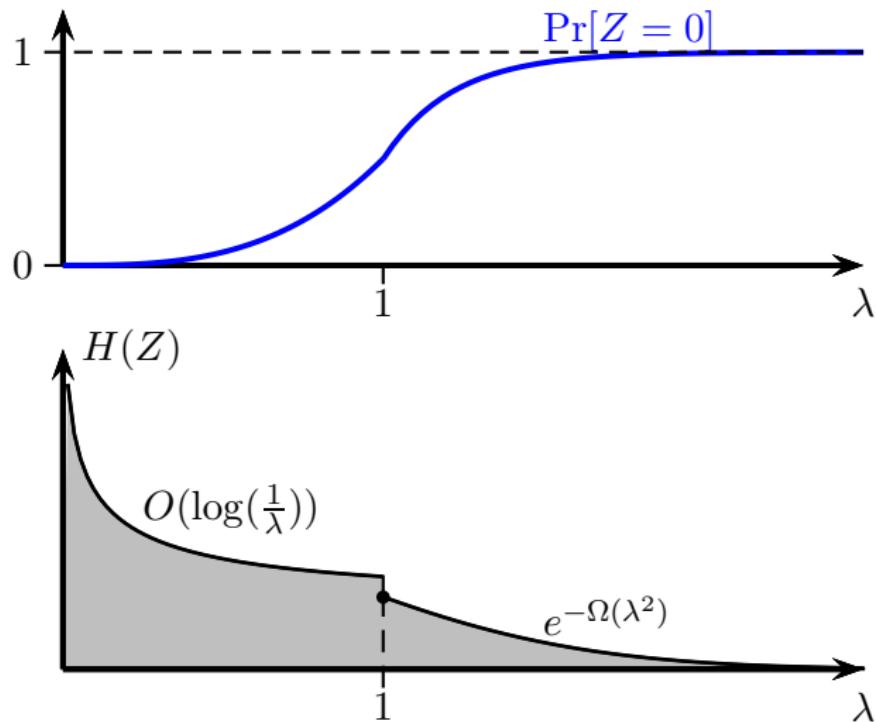
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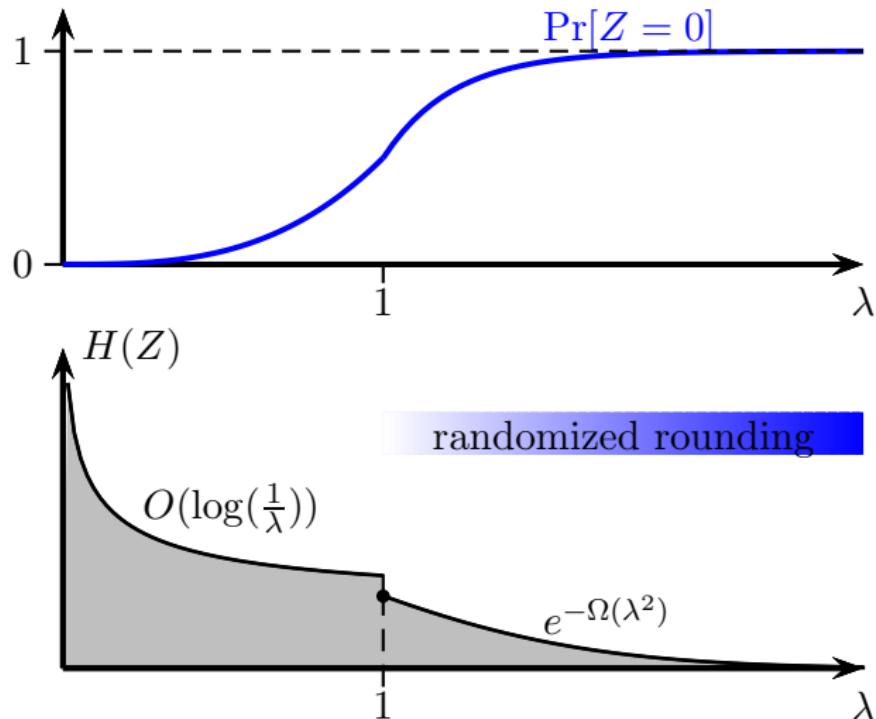
A bound on the entropy

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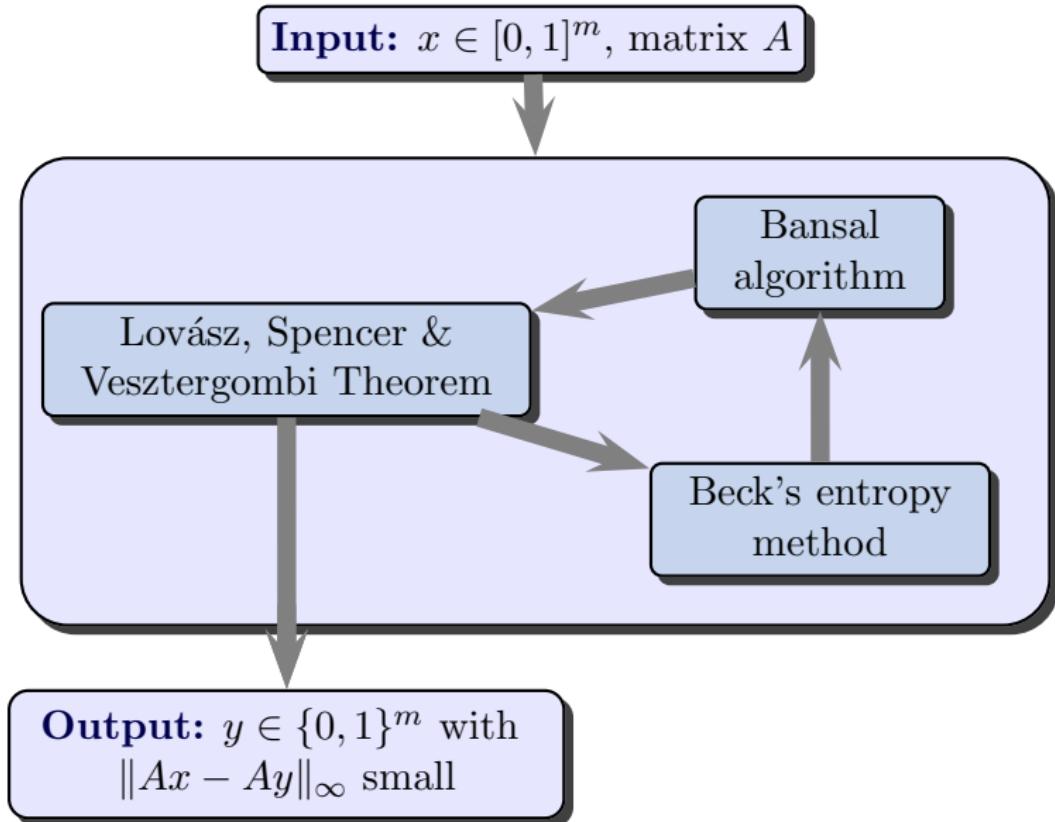


A bound on the entropy

- ▶ Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$.
- ▶ Consider $Z := \left\lceil \frac{\sum_j \chi(j) \cdot \alpha_j}{\lambda \cdot \|\alpha\|_2} \right\rceil$ with $\chi(j) \in \{\pm 1\}$ at random



A schematic view



A general rounding theorem

Theorem

Input:

- ▶ matrix $A \in [-1, 1]^{n \times m}$, $\Delta_i > 0$ satisfying entropy condition for all submatrices
- ▶ vector $x \in [0, 1]^m$
- ▶ row weights $w(i)$ ($\sum_i w(i) = 1$)

There is a random variable $y \in \{0, 1\}^m$ with

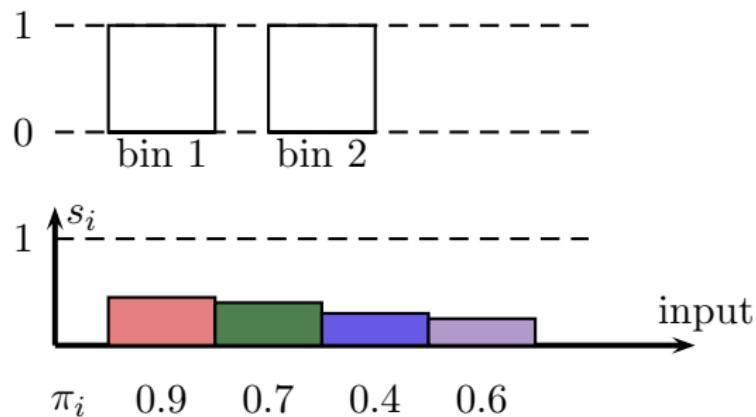
- ▶ *Bounded difference:*
 - ▶ $|A_i x - A_i y| \leq O(\log(m)) \cdot \Delta_i$
 - ▶ $|A_i x - A_i y| \leq O(\sqrt{1/w(i)})$
- ▶ *Preserved expectation:* $E[y_i] = x_i$.

Bin Packing With Rejection

Input:

- ▶ Items $i \in \{1, \dots, n\}$ with **size** $s_i \in [0, 1]$, and **rejection penalty** $\pi_i \in [0, 1]$

Goal: Pack or reject. Minimize # **bins** + rejection cost.

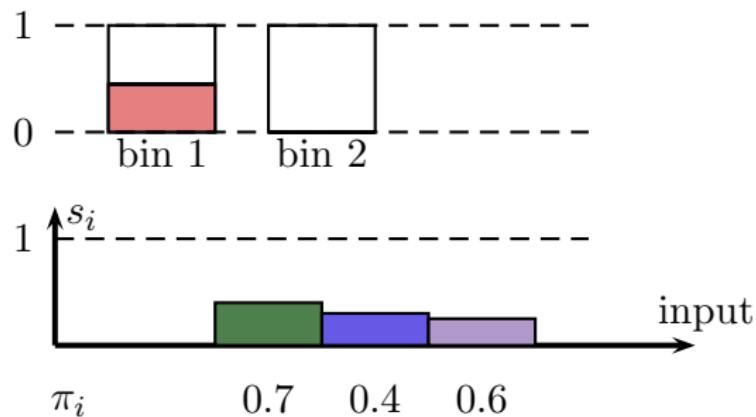


Bin Packing With Rejection

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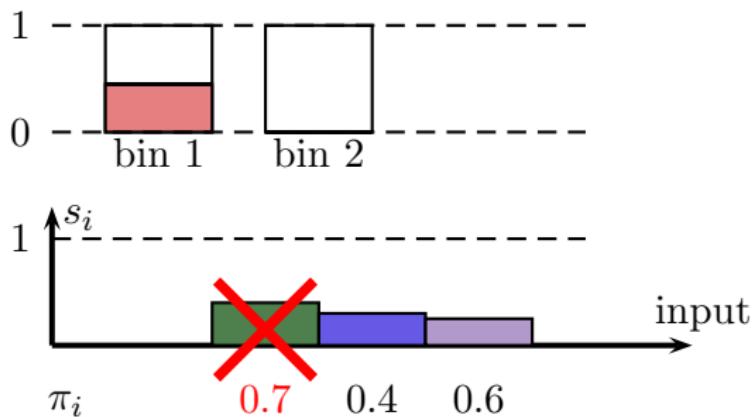


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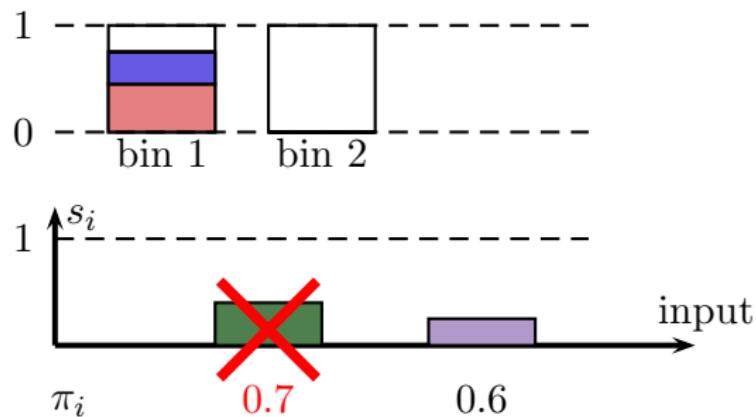


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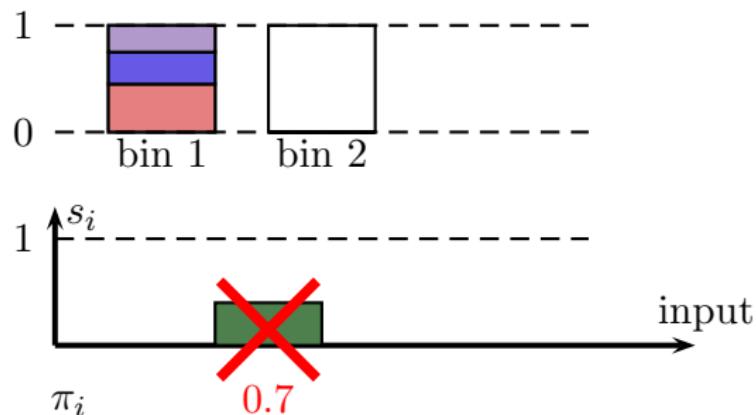


Bin Packing With Rejection

Input:

- Items $i \in \{1, \dots, n\}$ with **size** $s_i \in [0, 1]$, and **rejection penalty** $\pi_i \in [0, 1]$

Goal: Pack or reject. Minimize # **bins** + rejection cost.



The column-based LP

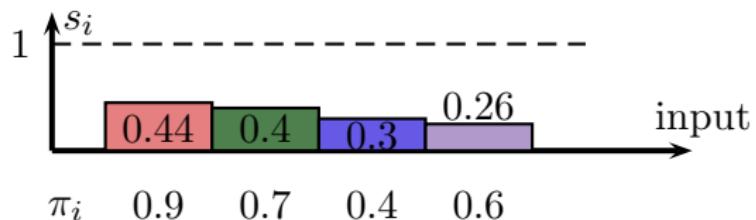
Set Cover formulation:

- ▶ **Bins:** Sets $S \subseteq [n]$ with $\sum_{i \in S} s_i \leq 1$ of cost $c(S) = 1$
- ▶ **Rejections:** Sets $S = \{i\}$ of cost $c(S) = \pi_i$

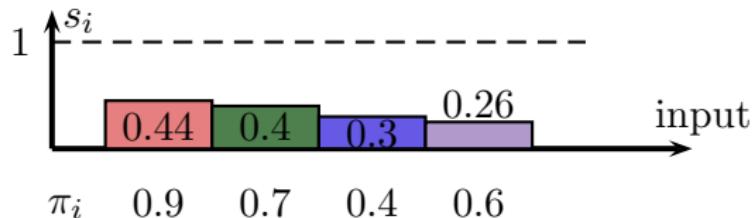
LP:

$$\begin{aligned} \min \sum_{S \in \mathcal{S}} c(S) \cdot x_S \\ \sum_{S \in \mathcal{S}} \mathbf{1}_S \cdot x_S &\geq \mathbf{1} \\ x_S &\geq 0 \quad \forall S \in \mathcal{S} \end{aligned}$$

The column-based LP - Example

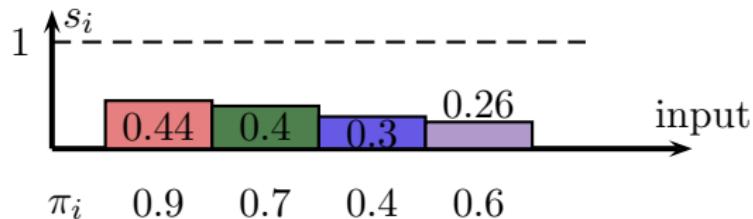


The column-based LP - Example



$$\begin{array}{l} \min (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ | \ .9 \ .7 \ .4 \ .6) \ x \\ \left(\begin{array}{cccccccccc|cccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{array} \right) x \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ x \geq \mathbf{0} \end{array}$$

The column-based LP - Example



$$\begin{array}{l}
 \min (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ | \ .9 \ .7 \ .4 \ .6) \ x \\
 \left(\begin{array}{cccccc|cccccc|cccc}
 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & | & 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & | & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & | & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & | & 0 & 0 & 0 & 1
 \end{array} \right) x \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
 \begin{array}{c}
 \uparrow \quad \uparrow \quad \uparrow \\
 1/2x \quad 1/2x \quad 1/2x
 \end{array} \\
 \begin{array}{c}
 \text{---} \\
 \text{---} \\
 \text{---}
 \end{array}
 \end{array}$$

A linear programming problem is shown. The objective function is $\min (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ | \ .9 \ .7 \ .4 \ .6) \ x$. The constraint matrix is a 4x12 matrix with entries colored by row: Row 1 (red) has 1s in columns 1-6 and 0s elsewhere; Row 2 (green) has 1s in columns 7-10 and 0s elsewhere; Row 3 (blue) has 1s in columns 11-12 and 0s elsewhere; Row 4 (purple) has 1s in columns 1-4 and 0s elsewhere. The right-hand side is a vector of 1s. The solution is given as $x \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Three arrows labeled $1/2x$ point from the bottom row to the first three columns of the constraint matrix, indicating that each component of the solution is half of the corresponding component in the vector.

Our results

Theorem

There is a randomized poly-time approximation algorithm for **Bin Packing With Rejection** with

$$APX \leq OPT_f + O(\log^2 OPT_f)$$

(with high probability).

- ▶ Previously best known: $APX \leq OPT_f + \frac{OPT_f}{(\log OPT_f)^{1-o(1)}}$ [Epstein & Levin '10]
- ▶ As good as classical **Bin Packing** [Karmarkar & Karp '82]

The end

Open question

Are there more (convincing) applications?

The end

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Thanks for your attention